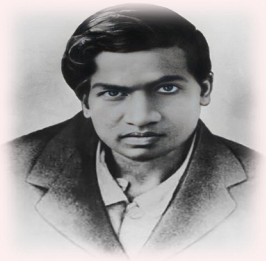
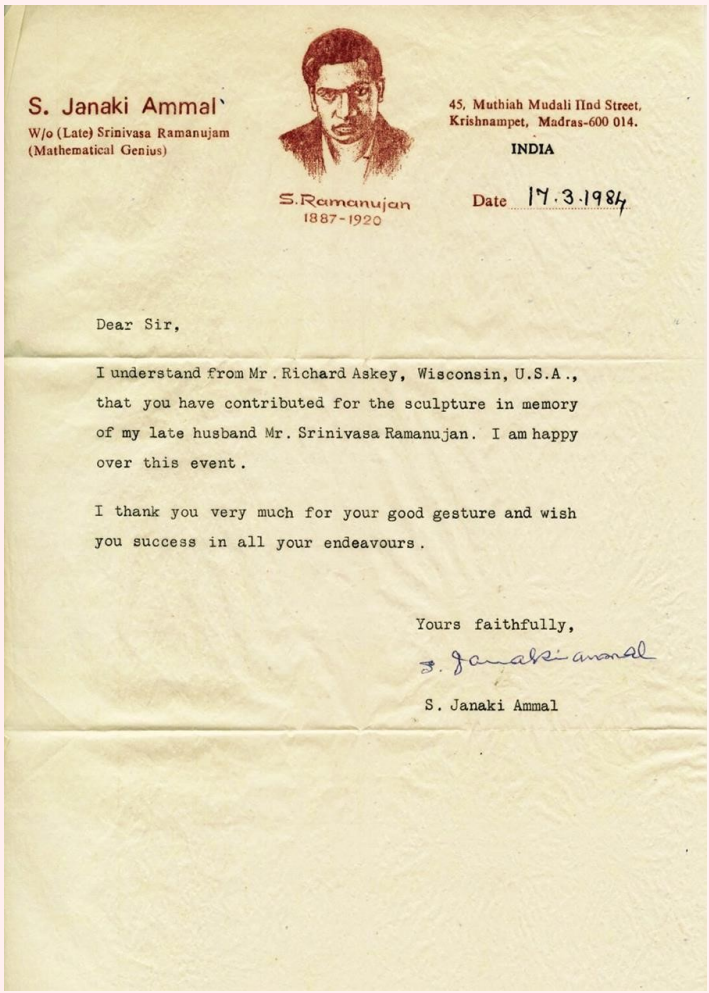
The Man who knew



Srinivasa

RAMANUJAN



S. Janaki Ammal
W/o (Late) Srinivasa Ramanujam
(Mathematical Genius)

S.Ramanujan
1887-1920

45, Muthiah Mudali IInd Street,
Krishnampet, Madras-600 014.
INDIA

Date 17.3.1984

Dear Sir,

I understand from Mr. Richard Askey, Wisconsin, U.S.A., that you have contributed for the sculpture in memory of my late husband Mr. Srinivasa Ramanujan. I am happy over this event.

I thank you very much for your good gesture and wish you success in all your endeavours.

Yours faithfully,

S. Janaki Ammal

S. Janaki Ammal

Department of Mathematics

Dr. M.G.R.
EDUCATIONAL AND RESEARCH INSTITUTE
DEEMED TO BE UNIVERSITY
University with Graded Autonomy Status
(An ISO 21001 : 2018 Certified Institution)
Periyar E.V.R. High Road, Maduravoyal, Chennai-95. Tamilnadu, India.

Srinivasa

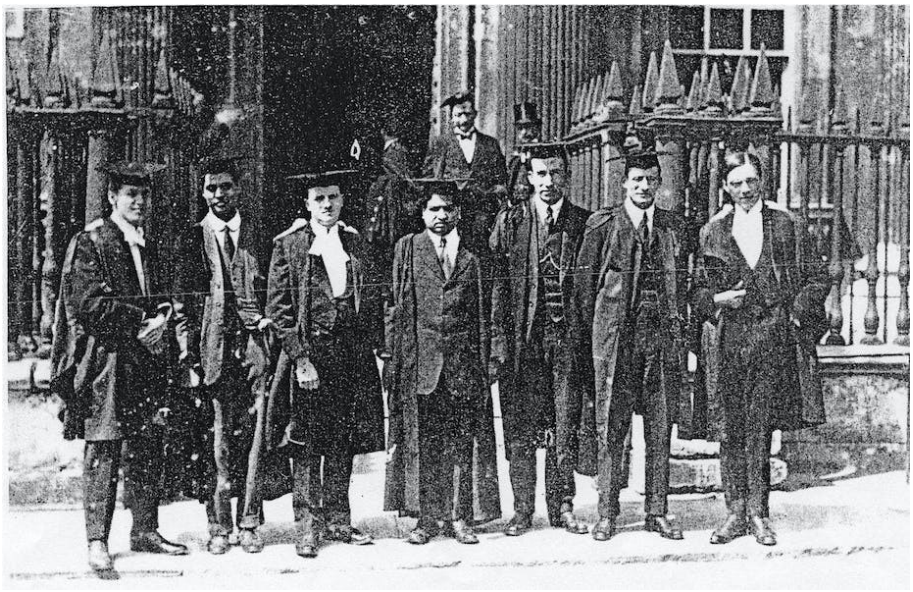
Ramanujan

The Man who knew INFINITY

Compiled by:

Dr S Ramalingam

Joint Registrar - Languages



Srinivasa Ramanujan (middle) with fellow scientists at Cambridge.

Department of Mathematics

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Srinivasa

Ramanujan

The Man who knew INFINITY

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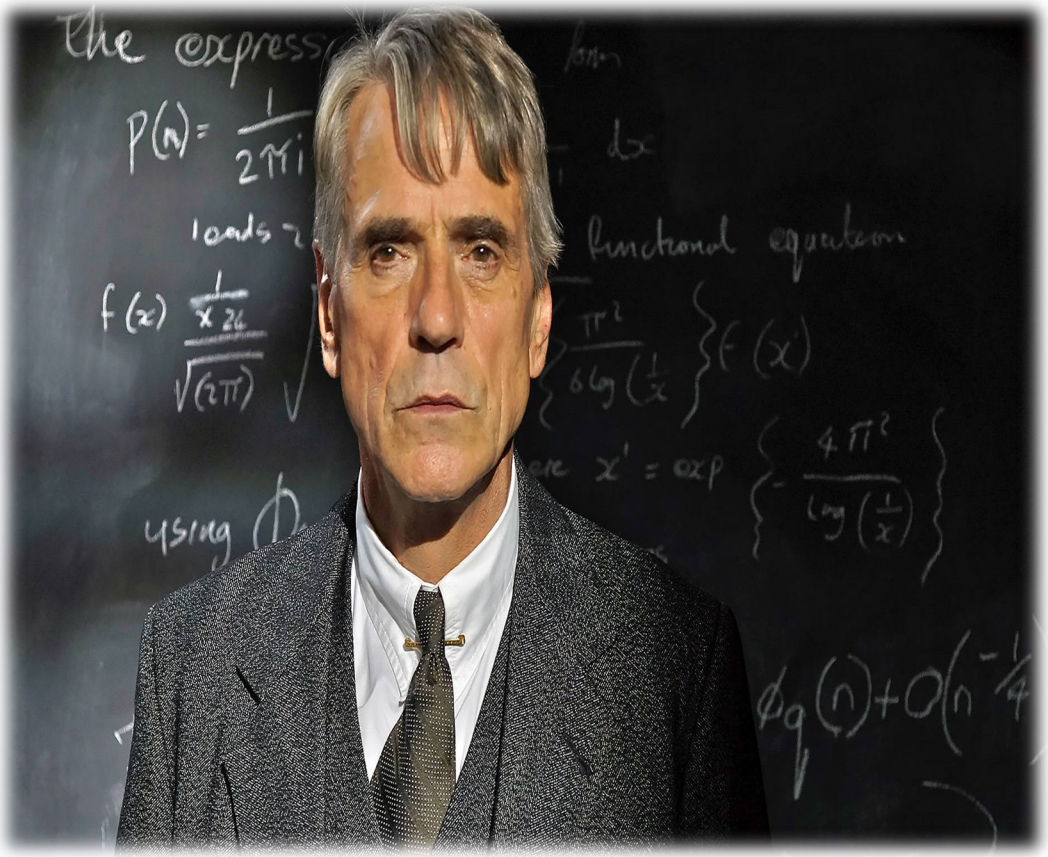
Godfrey Harold Hardy

He is perhaps even better known for his adoption and mentoring of the self-taught Indian mathematical genius, Srinivasa Ramanujan.



7 Feb 1877 <<*>> 1 Dec 1947

Foreword

Mathematics is very useful in everyday life. We use math concepts, as well as the skills we learn from practising math problems every day. Mathematics gives us a way to understand patterns, define relationships, and predict the future. It helps us do many important things in our daily lives.

Indian mathematics emerged in the Indian subcontinent from 1200 BCE until the end of the 18th century. In the classical period of Indian mathematics (400 CE to 1200 CE), important contributions were made by scholars like Aryabhata, Brahmagupta, Bhaskara II and Varāhamihira.

Mathematics has played a significant role in the development of Indian culture for millennia. Mathematical ideas that originated in the Indian subcontinent have had a profound impact on the world. Swami Vivekananda said: 'you know how many sciences had their origin in India. Mathematics began there. You are even today counting 1, 2, 3, etc. to zero, after Sanskrit figures, and you all know that algebra also originated in India.'

Ramanujan's contribution extends to mathematical fields such as complex analysis, number theory, infinite series, and continued fractions. Infinite series for pi: In 1914, Ramanujan found a formula for infinite series for pi, which forms the basis of many algorithms used today.

Srinivasa Ramanujan (1887-1920), the man who reshaped twentieth-century mathematics with his various contributions in several mathematical domains, including mathematical analysis, infinite series, continued fractions, number theory, and game theory is recognized as

one of history's greatest mathematicians. Leaving this world at the youthful age of 32, Ramanujan made significant contributions to mathematics that only a few others could match in their lifetime. Surprisingly, he never received any formal mathematics training. Most of his mathematical discoveries were based only on intuition and were ultimately proven correct. With its humble and sometimes difficult start, his life story is just as fascinating as his incredible work. In India every year from 2012, Ramanujan's birth anniversary on December 22 is observed as National Mathematics Day.

Srinivasa Ramanujan, the mathematical genius, came to be recognized only posthumously for his incredible contribution to the world of Mathematics. Leaving this world at the young age of 32, Srinivasa Ramanujan (1887-1920) contributed a great deal to mathematics that only a few could overtake in their lifetime.

To commemorate his Memorial Day on 26 April 2023, our Department of Mathematics is bringing out a book on him entitled – "*The Man how knew infinity Srinivasa RAMANUJAN*". The special feature of the book is that it contains two films on Srinivasa Ramanujan: (1) A 2015 British biographical drama film about the Indian mathematician Srinivasa Ramanujan, based on the 1991 book of the same name by Robert Kanigel: '***The Man Who Knew Infinity***'; and (2) A Tamil Full Movie '***Ramanujan***' (2014), a biographical film based on the life of Indian Mathematician Srinivasa Ramanujan. [with English sub-titles].

Chapter-17 provides the information about 200+ international famous mathematicians – their life, contributions, photos and the relevant web

links to know more about them. Chapter-7 discusses about Godfrey Herold Hardy, who is a mentor and guide to Ramanujan. Another chapter [Mathematical Pilgrimage] takes the readers to an exciting journey to all the places associated with the mathematical genius Srinivasa Ramanujan and offers a few interesting things about this beautiful mind and the dedication with which he lived his life.

Book also provides the detailed information on 'Ramanujan Museum', 'The Ramanujan Music Park', 'Srinivasa Ramanujan Centre' and also, the chapter on biography offers an interesting but a different version of his life and other details. The highly selected pictures and photos covering his life, family members, villages where he lived, his visit to Cambridge, his handwritten notes, national mathematics day, etc., provided in Chapter-22 throws an interesting pictographic world of Srinivasa Ramanujan.

Some important 24 web links - covering his life, organizations, his works, honours, his contributions – have been provided with the very useful annotations. I do hope that all the stake holders – students, teachers & scholars - would find this more useful in their academic venture.

I appreciate our department of mathematics in bringing out a very rare compendium during the 103rd death anniversary of Srinivasa Ramanujan.

14 April 2023

Prof Dr CB Palanivelu
Registrar
Dr MGR Educational and Research Institute
Maduravoyal, Chennai – 600 095

PREFACE

The Indus valley civilization is considered to have existed around 3000 BCE. Two of its most famous cities, Harappa and Mohenjo-Daro, provide evidence that construction of buildings followed a standardized measurement which was decimal in nature.

One of the greatest mathematicians of all time, Srinivasa Ramanujan was born in 1887 in the Southern part of India. He is still remembered for his contributions to the field of mathematics. Theorems formulated by him are to date studied by students across the world and within very few years of his lifespan, he made some exceptional discoveries in mathematics.

Ramanujan initially developed his own mathematical research in isolation. According to Hans Eysenck, "he tried to interest the leading professional mathematicians in his work, but failed for the most part. What he had to show them was too novel, too unfamiliar, and additionally presented in unusual ways; they could not be bothered".

Although Ramanujan had no official training in mathematics, even then, he was able to solve problems that were considered unsolvable. He published his first paper in 1911. In January 1913, Ramanujan began a postal conversation with an English mathematician, G.H. Hardy at

the University of Cambridge, England and wrote a letter after having seen a copy of his book Orders of infinity. He found Ramanujan's work to be extraordinary and arranged for him to travel to Cambridge in 1914. As Ramanujan was an orthodox Brahmin, a vegetarian, his religion might have restricted him to travel. This difficulty of Ramanujan was solved partly by E H Neville, a colleague of Hardy. Hardy after analysing the works of Ramanujan, said, "Ramanujan had produced ground-breaking new theorems, including some that defeated me completely. I had never seen anything in the least like them before.'

During his short life, Ramanujan independently compiled nearly 3,900 results (mostly identities and equations). Many were completely novel; his original and highly unconventional results, such as the Ramanujan prime, the Ramanujan theta function, partition formulae and mock theta functions, have opened entire new areas of work and inspired a vast amount of further research. Of his thousands of results, all but a dozen or two have now been proven correct. *The Ramanujan Journal*, a scientific journal, was established to publish work in all areas of mathematics influenced by Ramanujan, and his notebooks—containing summaries of his published and unpublished results—have been analysed and studied for decades since his death as a source of new mathematical ideas.

As late as 2012, researchers continued to discover that mere comments in his writings about "simple properties" and "similar outputs" for certain findings were themselves profound and subtle. He became one of the youngest Fellows of the Royal Society and only the second Indian member, and the first Indian to be elected a Fellow of Trinity College, Cambridge. Of his original letters, Hardy stated that a single look was enough to show they could have been written only by a mathematician of the highest calibre, comparing Ramanujan to mathematical geniuses such as Euler and Jacobi.

The idea of bringing a book was conceived during Ramanujan's birthday anniversary on 22 Dec 2022. After a sincere and serious brainstorming session, it is decided to focus mainly on his life and contributions to mathematics. Two important features included here are: (1) 2015 British Film '*The Man Who Knew Infinity*' and (2) *2014 Tamil Film 'Ramanujan'.*

Other interesting features are: (a) Mathematical Pilgrimage, (b) His mentor Hardy, (c) National Mathematics Day (d) his contributions of mathematics, (e) his wife Janaki Ammal, (f) Birthday Magic Square, (g) Srinivasa Ramanujan's Notes, (h) Ramanujan Museum at Chennai (i) details of 200 international mathematicians, (j) his typical biography, (k) his unpublished works and (l) Photos/pictures covering his life, living places, his family members, his to visit Cambridge, etc.,

Some of the relevant/suitable information/details have been culled from the most reliable sources for the purpose of enriching the contents of some of the chapters in the book. Wherever possible, the necessary sources are cited and also, to enable the readers to visit sources the respective web links of the sources are provided. But during the process, it is learnt that without providing the web links the task may not be complete and comprehensive one. In addition to web links, a separate chapter giving 24 web links with exhaustive annotations is given. Some of the papers on Ramanujan written by famous international mathematicians are added as annexures at the end of the book.

We hope that the students, scholars and other stake holders in mathematics would find the book is very useful and informative.

Prof Dr S Ramalingam
Joint Registrar – Languages
{Author}

24 April 2023
Chennai – 600 095

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https://www.britannica.com/biography/Srinivasa-Ramanujan

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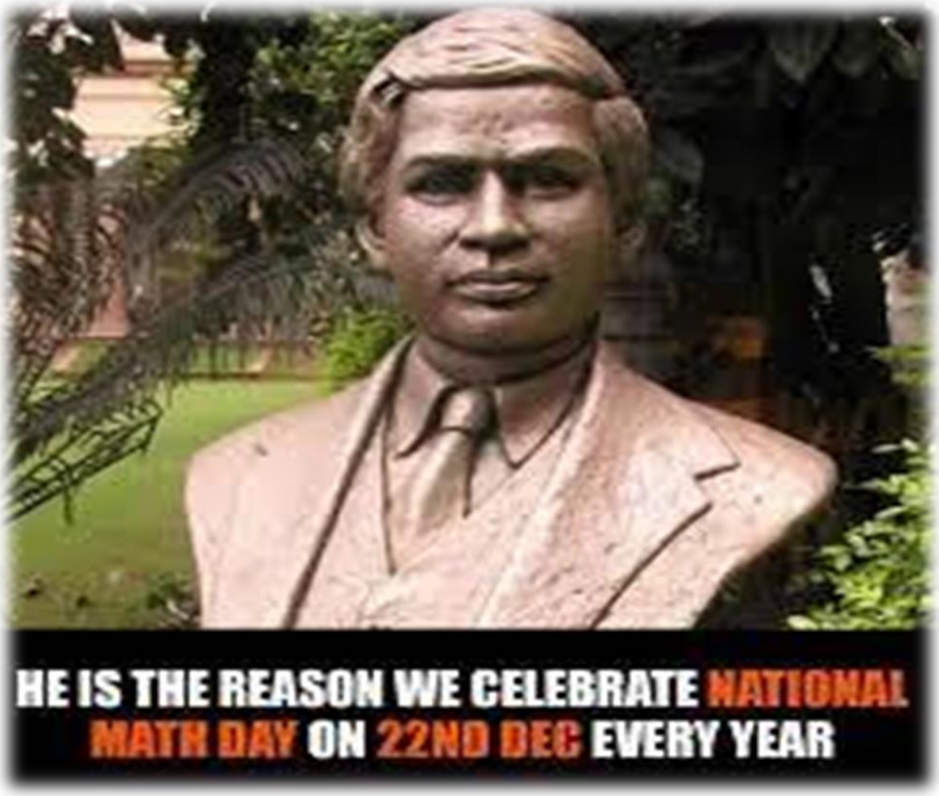
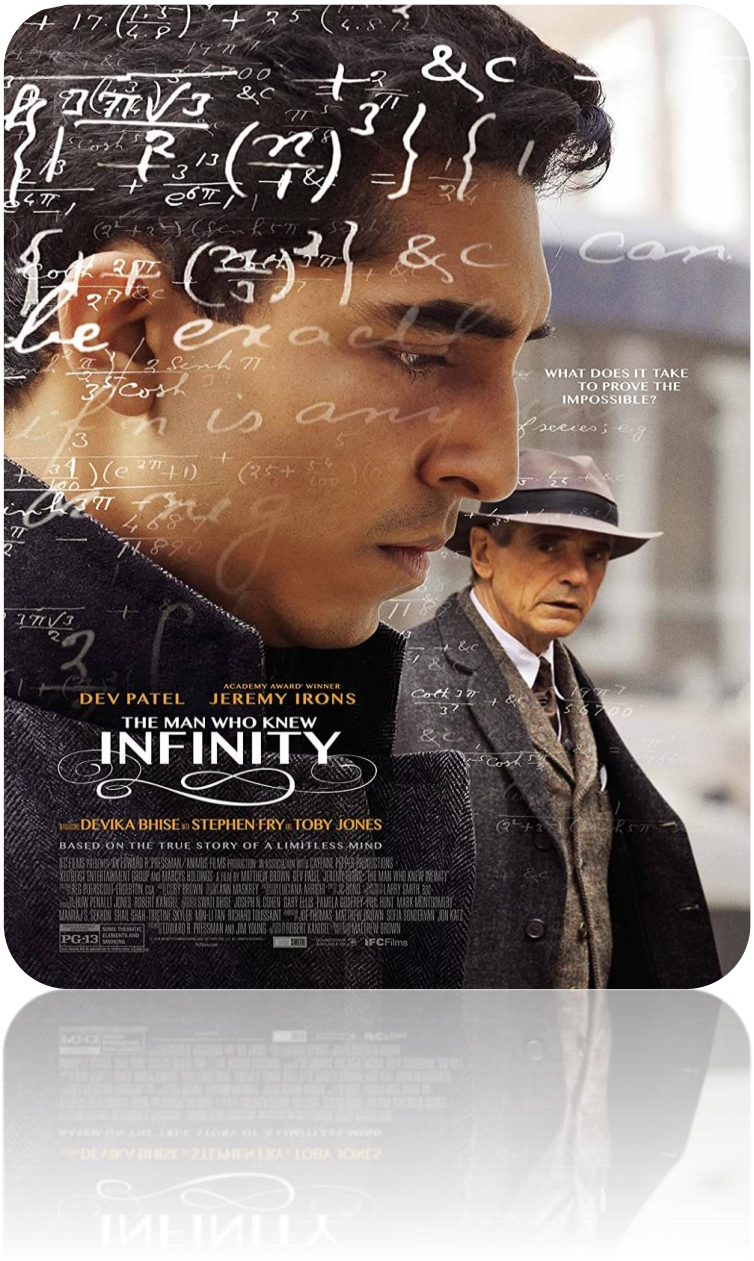
Ramanujan's House



The Man Who Knew Infinity


WHAT DOES IT TAKE TO PROVE THE IMPOSSIBLE?
ACADEMY AWARD® WINNER
DEV PATEL JEREMY IRONS
THE MAN WHO KNEW
INFINITY
DEVIKA BHISE STEPHEN FRY TOBY JONES
BASED ON THE TRUE STORY OF A LIMITLESS MIND
PG-13
IFC Films

Visit to watch the Film: The Man Who Knew Infinity

https://www.imdb.com/title/tt0787524/

Directed by

Matt Brown ... (as Matthew Brown)

Writing Credits (in alphabetical order)

Matt Brown ... (screenplay)

Matt Brown

Robert Kanigel ... (biography)

Cast (in credits order) verified as complete

	Jeremy Irons	...	G.H. Hardy
	Dev Patel	...	S. Ramanujan
	Malcolm Sinclair	...	Professor Cartwright
	Raghuvir Joshi	...	Narasimha
	Dhritiman Chatterjee	...	Narayana Iyer (as Dhritiman Chaterji)
	Stephen Fry	...	Sir Francis Spring
	Arundathi Nag	...	Komalatammal
	Devika Bhise	...	Janaki
	Pádraic Delaney	...	Beglan
	Toby Jones	...	John Edensor Littlewood
	Jeremy Northam	...	Bertrand Russell

	San Shella	...	Dr. Muthu
	Richard Cunningham	...	Hobson
	Thomas Bewley	...	Baker
	Anthony Calf	...	Robert Alfred Herman
	Richard Johnson	...	Vice Master Henry Jackson
	Kevin McNally	...	Major Percy Alexander MacMahon (as Kevin R. McNally)
	Pip Barclay	...	Student
	David Shaw Parker	...	Porter (as David Shaw-Parker)
	Dominic Cazenove	...	Waiter
	Shazad Latif	...	Chandra Mahalanobis
	Nicholas Agnew	...	Andrew Hartley
	Roger Narayan	...	Scribe
	Alan Booty	...	Grocer
	Shenagh Govan	...	Postal Worker
	Alexander Forsyth	...	Cadet
	Enzo Cilenti	...	Doctor
	Elaine Caulfield	...	Hospital Nurse

	Alex Bartram	...	Hospital Doctor
	Christopher Ravenscroft	...	J.J. Thompson

Produced by

Nick Bain	...	associate producer
Manjul Bhargava	...	associate producer
Swati Bhise	...	associate producer: India / executive producer
Matt Brown	...	producer (as Matthew Brown)
Audrey Kathleen Chong	...	associate producer
David Cochrane	...	associate producer
Joseph Newton Cohen	...	executive producer (as Joseph N. Cohen)
Harsh Dave	...	line producer: India
Meeta Dave	...	line producer: India
Alice Dawson	...	line producer: India
Gary Ellis	...	executive producer
Bruce Wayne Gillies	...	line producer: pick-up unit
Melissa Robyn Glassman	...	associate producer
Pamela Godfrey	...	executive producer
Phil Hunt	...	executive producer
Huw Penallt Jones	...	co-producer
Robert Kanigel	...	co-producer
Jon Katz	...	producer
Nariaki Kumagi	...	associate producer
Vinod Kumar	...	associate producer
Tayyab Madni	...	associate producer
Takase Masayuki	...	associate producer
Mark Montgomery	...	executive producer
Anthony Timothy Morris	...	associate producer
Simon Moseley	...	line producer
Toshinori Nakai	...	associate producer
Ken Ono	...	associate producer
James Passin	...	associate producer
Edward R. Pressman	...	producer
Sarah Ramey	...	associate producer
Compton Ross	...	executive producer
Manraj S. Sekhon	...	executive producer
Shail Shah	...	executive producer
Parveen Shereef	...	associate producer
Tristine Skyler	...	executive producer
Sofia Sondervan	...	producer
Apparswamy Subramanian	...	associate producer
Min-Li Tan	...	executive producer

Masaaki Tanaka	...	executive producer
Anand Tharmaratnam	...	associate producer
Jomon Thomas	...	producer (as Joe Thomas)
Richard Toussaint	...	executive producer
Sam Raj Kumar Wilson	...	associate producer
Jim Young	...	producer

Music by

Coby Brown

Cinematography by

Larry Smith ... director of photography

Film Editing by

JC Bond

Casting By

Seher Latif ... (India)

Reg Poerscout-Edgerton

Production Design by

Luciana Arrighi

Rajeevan ... (as Rajeevan Nambiar)

Art Direction by

Andrew Munro ... supervising art director

Justin Warburton-Brown

Set Decoration by

Liz Ainley

Costume Design by

Ann Maskrey

Makeup Department

Louise Coles	...	hair stylist / makeup artist
Hannah Edwards	...	daily hair stylist / daily makeup artist
Nalini Fernandes	...	makeup artist: India (as Nalini Fernandez)
Annette Field	...	hair stylist: Jeremy Irons / makeup artist: Jeremy Irons
Cate Hall	...	hair designer / makeup designer
Simmi Shanmughan	...	assistant makeup artist: India
Indradeo Thakur	...	assistant makeup artist: India
Charlotte Wood	...	hair trainee / makeup trainee

Production Management

Meg Clark	...	post-production supervisor
John David Gunkle	...	unit manager
Vibhu Neelmani Kant Mishra	...	production assistant
Katie Shahrokh	...	post-production manager
Samantha Waite	...	production manager

Second Unit Director or Assistant Director

Raj Acharya	...	first assistant director
Livia Aranha	...	second assistant director
Lydia Currie	...	first assistant director
Jennie Fava	...	second assistant director
Grace Hartley	...	third assistant director
Laura Hunter	...	additional assistant director (as Laura Wootton)
Sean Poovaiya Kongettira	...	second unit director
Dhruv Piyush Panjuani	...	director's assistant
Dev Rohira	...	assistant director
Liam Shaw	...	third assistant director

Art Department

Joe Bovington	...	props
Toby Browne	...	props
Liz Colbert	...	graphic designer
Celia De La Hey	...	art department researcher
Hudstin Fernandez	...	graphic designer
Hugh Fottrell	...	props
Adrian Greenwood	...	production buyer
Charlotte Hutchings	...	art department runner
Lizzie Kilham	...	standby art director
Max Klaentschi	...	art department assistant
Veronique Langdon	...	dressing props
Naomi Leigh	...	petty cash buyer
Richard MacMillan	...	props
Laura Miller	...	art department assistant

Paul Mitchell ... chargehand set dresser

John Moolenschot ... construction manager

Daisy Moseley ... set dresser

Helen Negus ... assistant graphic designer

Declan O'Brien ... art department experience

David Pearce ... scenic painter

Ryan Saward ... standby

Rebecca Slade ... petty cash buyer

Gareth Sprey ... art department work experience

James Waddingham ... h.o.d. carpenter

Damian Léon Watts ... assistant art director

Arthur Wicks ... property master

Matthew Parsons ... scenic painter (uncredited)

Sound Department

Simon Bysshe ... production sound mixer: second unit

Tim Cavagin ... re-recording mixer

Simon Chase ... dialogue editor

Rob Chen ... sound editor (as Robert Chen)

Nicholas Cochran ... adr recordist (as Nick Cochran)

Robert Dehn ... sound studio manager

Mark DeSimone ... adr mixer

Samir Foco ... sound designer

Detlef Halaski ... re-recording mixer

Neil Hillman ... adr mixer

Ben Jeffes ... second assistant sound

Phil Lee ... sound effects mixer

Travis MacKay ... a.d.r. mixer

Adam Mendez ... foley mixer

Nigel Mills ... additional dialogue editor

Chris Navarro ... adr mixer

Lucas Pimentel ... assistant sound editor

Amala Popuri ... production sound mixer: India

Will Riley ... additional re-recording mixer / additional re-rerecording mixer

Sai Shravanam ... sound recordist

Dario Swade ... foley editor

Dylan Voigt ... sound maintenance

Ian Voigt ... sound mixer

Max Walsh ... sound mix technician

Special Effects by

Jonathan Bullock ... special effects senior tech

Neil Damman ... special effects technician

Richard Van Den Bergh ... special effects supervisor

Visual Effects by

Svilen Aynadzhiev ... digital compositor

Tiago Faria ... rotoscoping artist

Andre Giordani ... matchmove artist 3D generalist: Prime Focus

Piers Hampton ... visual effects supervisor

John Hardwick ... digital compositor

Will Hardwick ... digital compositor

Richard Klein ... lead fx artist

Bryan Servante ... vfx generalist

Stephanie Jean Staunton ... digital compositor

Stunts

Paul Scruff Bates ... safety diver

Crispin Layfield ... stunt coordinator

Andy Piers Morris ... dive supervisor

Camera and Electrical Department

Robert Binnall ... director of photography: second unit

Richard Blanshard ... epk stills photographer / still photographer

Billy Clifford ... practical electrician

Paul Dain ... first assistant "b" camera dailies

Mardon de Carvalho ... digital imaging technician

Kevin Foy ... grip: "b" camera/key grip

Dan Gamble ... "a" camera second assistant camera: main unit

Ganesh Hegde ... gaffer

Ganesh s Hegde ... gaffer

Ryan Huffer ... electrician

Andrew Jones ... second assistant camera: "b" camera

Jack Harrison Knott ... camera trainee

Siba Prasad Maity ... second assistant camera: a camera india

Kev Marchant ... grip trainee: dailies

Michael McDermott ... gaffer

Ray Meere ... first assistant camera dailies

Stefan Mitchell ... lighting technician

Ronan Murphy ... key grip

Larry Park ... rigging gaffer

Sanket Pawar ... d.i.t

Jack Powell ... electrician / lighting technician

Sreeram Ramanathan ... digital imaging technician: India schedule

Oliver Squire ... camera trainee

Steve Wells ... grip: trainee

Paul Wheeldon ... first assistant camera

Casting Department

Vanessa Baker ... adr voice casting

Katy Covell ... casting assistant

Ollie Gilbert ... casting assistant

Lillie Jeffrey ... casting associate

Trishaan ... casting associate (as Trishaan Sarkar) / casting associate: India (as Trishaan Sarkar)

Costume and Wardrobe Department

Michael Agosta ... key costumer

Anna Beckett ... costume daily

Anne T. Delaney ... costume: dailies

Daisy Marcuzzi ... costume standby

Heidi McQueen-Prentice ... costume daily

Nicola Rapley ... costume supervisor

Saehee Simmons ... costume standby

Alice Speak ... costume standby

Ann Taylor ... assistant costume designer

Editorial Department

Drew Armstrong ... i/o data operator

Herbert Butler ... on-line editor

Tom Cairns ... digital intermediate conform editor

Michelle Cort ... digital intermediate conform editors

Martin Critchley ... data i/o: Molinare

Kelvin Daley ... digital intermediate on-line assistant

Johnathan Wei Dickinson ... conform editor

Leah Foster ... post-production assistant

Ollie Gill ... digital intermediate assistant

Elyse Holloway ... first assistant editor

Alice Jowitt ... data transfer

Steve Knight ... digital intermediate coordinator

Roisin Md ... flame artist & on-line / on-line editor

Jonny Monks ... data transfer / vt operator

Lizzie Newsham ... digital intermediate conform editor

Gareth Parry ... on-line editor

Jateen Patel ... digital colourist

Dave Rose ... on-line editor

Marie Valentino ... senior digital intermediate producer

Tim Waller ... senior colorist

Isabel Zippert ... data i/o

Location Management

Chris Barnett ... location assistant

Ben Gladstone ... location manager

Jojo Warne ... assistant location manager

Jessica Woodland ... assistant location manager

Music Department

Charles M. Barsamian ... executive music producer

Thomas Bergersen ... composer: trailer music

Tony Blondal ... orchestrator

Coby Brown ... composer: theme music

Peter Clarke ... music editor

Stephen Coleman ... orchestrator

Richard Henderson ... additional music editor: Temp - uncredited

Paul Koch ... music programming / technical score advisor

Tami Lester ... music supervisor

Bhavani Prasad ... veena

Andy Ross ... music producer

Saroja ... musician: dilruba

Sai Shravanam ... Indian Music Producer: additional music / recording/scor engineer -resound india studios / tabla/surmandal/tanpura

Del Spiva ... music editor

Justin Stanley ... composer: additional music / mixed by

Ganapathi Subramanian ... mridangam/ghatam/kanjira/side percussion

N. Sundar ... musician: morsing (as Sundar N.)

R.K. Sundar ... assistant recording engineer

Keerthana Vaidyanathan ... vocalist: Indian

Vishnu Vijay ... flute

Script and Continuity Department

Roxanne Cuenca ... assistant script supervisor/second unit script supervisor

San Davey ... script supervisor

Transportation Department

Matthew Ballard ... transportation coordinator: pickup unit

Tony Blyth ... unit driver

Nick Bramston	...	unit driver
Glenn Charter	...	unit driver
Kamil Cieslak	...	facilities account manager
Michael Geary	...	action vehicle supplier: Motorhouse
Barrie Williams	...	transportation captain
Nigel Willson	...	driver minibus

Additional Crew

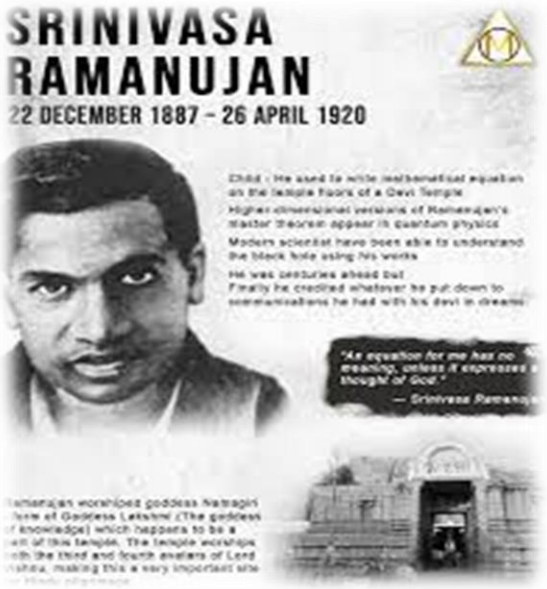
Glyn Alderson	...	Satellite Internet Wifi Engineer & Wifi Support
Swati Bhise	...	India Cultural Advisor / India cultural consultant / Indian cultural advisor
Faz Buffery	...	floor runner
Sorel Carradine	...	assistant to director
Karen Chiarello	...	advisor: astrology
Peter Dally	...	legal advice
Charles Edwards	...	clearances coordinator
Laura Evans	...	production secretary
Johan Ewing	...	legal executive
Amanda Fox	...	assistant accountant
Stephen Gallacher	...	floor runner: dailies
Bill Grantham	...	finance legal: Ariel Film Partners
Jonny Hodgetts	...	floor runner: dailies uk
Aaron Hopkins	...	production assistant
Christopher G. Howland	...	production finance
Raghuvir Joshi	...	dialog coach
Hilary Joyce	...	accounts assistant
Trisha Joyce	...	unit medic
Sumeet Kamath	...	workflow assistant: India schedule
Sarah Louise Kay	...	production accountant
Hannah Leader	...	legal advice
Fernando Lueches	...	marketing and communications: Spain
Kelly McKee	...	assistant: Mr. Pressman

Abby Mills	...	assistant production coordinator
Andy Piers Morris	...	safety diver
Maggie Murray	...	payroll accountant
Wolf Newton	...	production assistant (as Wolf J Newton) / production assistant
Ken Ono	...	technical advisor: mathematics
Tom Ormerod	...	assistant production coordinator
David Orpheus	...	crowd production assistant
Emily Precious	...	assistant to producers
Sam Pressman	...	director: Making of (as Sammy Jack Pressman)
Sreeram Ramanathan	...	workflow consultant: India schedule
Luke Randolph	...	completion guarantor: for European Film Bonds
Katy Riddell	...	assistant: Mr. Irons
Sarmad Saleh	...	legal advice
Christine Samways	...	assistant accountant
Kumar Santosh	...	production accountant
Joan Scheckel	...	workshop development
Graham Walters	...	caterer: craft service
Sally Webb	...	payroll accountant
Victoria Zalin	...	production coordinator
Javier Alonso Cerdeiriña	...	daily runner (uncredited)
Alessia Bologna	...	runner (uncredited)
Kriss TheDish Brown	...	on-set internet engineer (uncredited)
Huma Khan	...	director legal post production (uncredited)

On a Mathematical Pilgrimage

Resource:

https://www.parentcircle.com/on-a-mathematical-pilgrimage-with-srinivasa-ramanujan/article



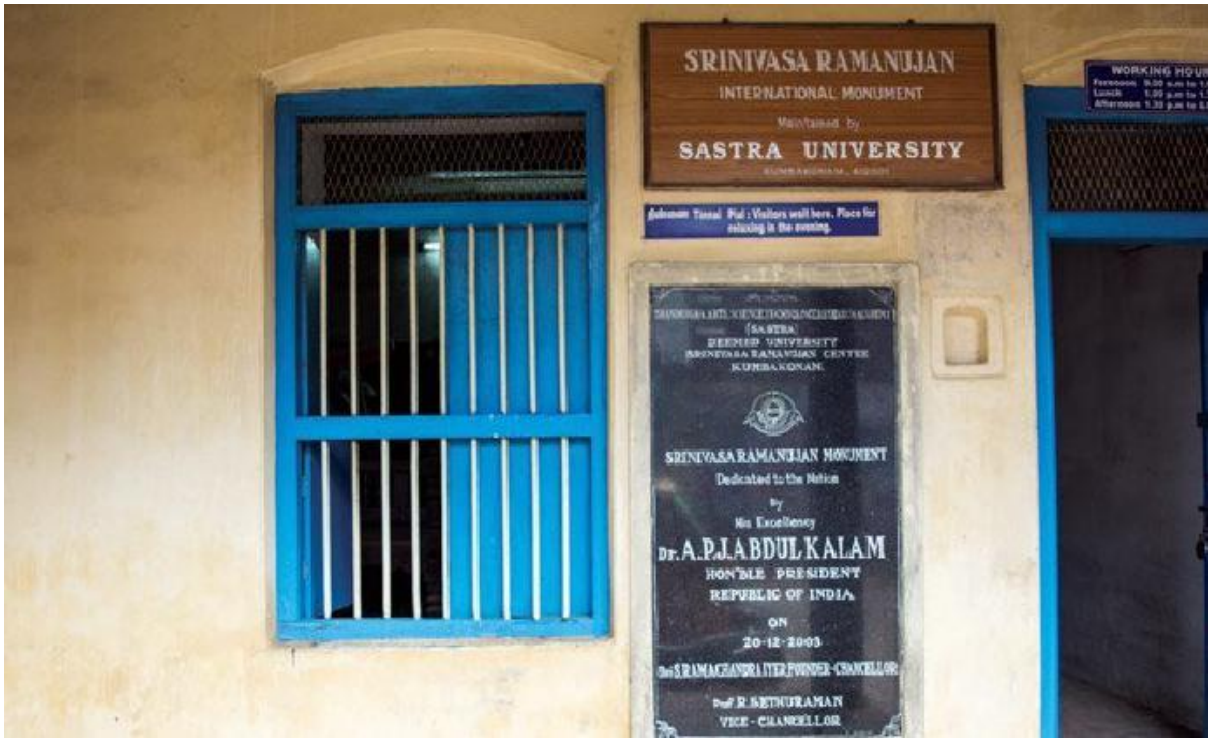
Let us have an exciting journey to all the places associated with the mathematical genius Srinivasa Ramanujan and learn a few interesting things about this beautiful mind and the dedication with which he lived his life.

Srinivasa Ramanujan was the greatest Mathematical mind to emerge from India in the last century and one of the brightest of all time. His findings took the mathematical world of his day by storm and his papers are being studied and researched even today. An autodidact and a genius, he compiled thousands of mathematical formulae in his short lifespan and was compared to great Mathematicians like Jacobi and Euler by his colleagues in Cambridge.

The life of Srinivasa Ramanujan, the great Indian Mathematician, was one who lived in deeds and not in years. In his short lifespan of 32 years, Ramanujan managed to leave behind a legacy of a vast body of formulations and theorems, that have immensely contributed to the subject, earning him international acclaim in the field of Mathematics.

A few places spread across Kumbakonam and Chennai (then Madras) in Tamil Nadu have preserved their connection to this beautiful mind.

What better way to teach life lessons to children than by taking them on a trip to places associated with the revered genius. Join us on a tour of these little-known places.



Srinivasa Ramanujan International Monument, Kumbakonam

Ramanujan's Ancestral Home

Born in Erode in 1887, Ramanujan was brought to the temple town of Kumbakonam by his parents when he was just a year old. Here, he spent his childhood and the early days of his youth. The house he lived in, which is just a stone's throw away from the Sarangapani temple, is now a museum with the status of an international monument, maintained by the deemed SASTRA (Shanmugha Arts, Science, Technology and Research Academy) University.

Step inside this ancestral property and you are sure to be overwhelmed by a sense of mysticism, an attribute that has often been assigned to Ramanujan's brand of Mathematics. A collection of old photographs and copies of theorems scribbled by him gives you a glimpse into the events from his life. The cot he slept on can still be found in one of the rooms. As you spend more time here, you cannot help but be touched by a haunting sense of time unchanged and the presence of a pure mathematical spirit.

How to get there: Kumbakonam is located about 300 km from Chennai and can be reached by rail or road. You can easily walk the distance from both the railway station and the bus terminus to Ramanujan's ancestral home.

The Alma Mater

Town Higher Secondary School, Kumbakonam

[The school is located less than a km from Ramanujan's ancestral home.]

After completing his primary schooling at Kangayan Primary School, Ramanujan enrolled at the Town Higher Secondary School in 1898. It was here that his Mathematical acumen began to take shape.

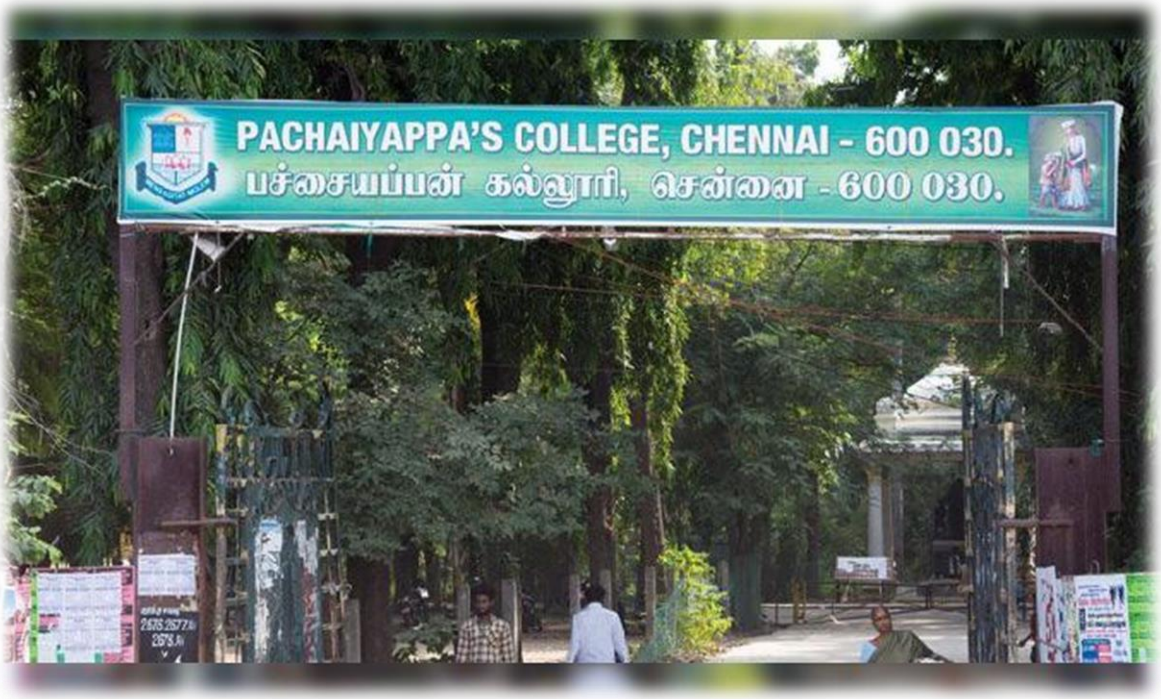
It is now popular history that when he was here, he laid his hands on a book called Synopsis of Elementary Results in Pure Mathematics by G S Carr in 1903. That was a watershed moment in the life of the genius, for it opened his line of thinking in the direction of original Mathematical theorization. He began to explore Mathematics by himself.

He passed out from the school in 1904 with distinction. The 152-year-old institution remains one of the leading schools of this quaint town.

In 1906, Ramanujan came to Madras (now Chennai) to find a foothold in life. He joined Pachaiyappas college in Shenoy Nagar to gain a Fellow of Arts degree. But

he didn't do well in his studies and failed in all the subjects except Mathematics.

Even in the subject he loved, he is said to have passed with low marks. This is believed to have been due to his passion and obsession with only those areas of the subject that interested him. Due to poor results, he had to drop out of college.



The next destination on this Mathematical trail is Triplicane in Chennai. Near the famous Arulmigu Sri Parthasarathy Swamy temple in Triplicane, a house named Gautamasrama, has history written all over it.

One of the many connections of this 146-year-old building is the wing that was rented by Ramanujan, where he stayed with his family after marrying Janaki Ammal in 1909. While he stayed here, Ramanujan worked out his theorems day in and day out, using not paper and pen, but a slate and chalk.

Despite the poverty that he faced and other difficulties, several events took place in his life here, which established him as one of the greatest mathematical minds of the world.

The most significant of them was the spotting of his talents by the famous English mathematician G H Hardy, who invited him to London. This laid the foundation for Ramanujan's recognition in the international arena.

The house is located around 4 km from the Chennai Central Railway Station.

Ramanujan Museum and Math Education Centre, Royapuram, Chennai

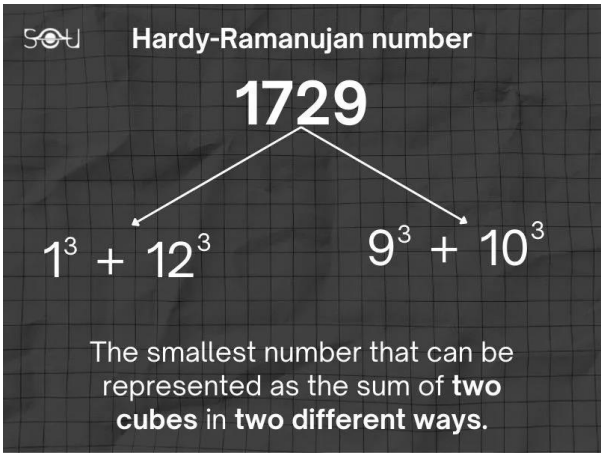
To complete the trip, your next destination is the Ramanujan Museum and Math Education Centre, in Royapuram, Chennai. Here again, you get a chance to catch

up with the Mathematician's life through a collection of books, photographs and exhibits.

The center houses an auditorium and a library. Regular workshops and classes are conducted for children of all age groups based on specific concepts in Mathematics. The center has been playing an active role in promoting innovations in the Math curriculum and the concept of a Mathematics lab to help children learn the subject with greater zeal.

The President dedicated the House of Ramanujan containing life and works of the prodigy to the Nation.

Ramanujan Number


Hardy-Ramanujan number
1729
$1^3 + 12^3$
$9^3 + 10^3$
The smallest number that can be represented as the sum of two cubes in two different ways.

Born in Tamil Nadu's Erode on December 22, 1887, the self-taught mathematical prodigy Srinivasa Ramanujan made extraordinary contributions to mathematical analysis, number theory, infinite series and continued fractions.

He was fascinated by numbers and made some remarkable contributions to the *partitio numerorum* branch of mathematics that deals with the study of partitions of numbers. The mathematical genius compiled approximately 3900 results independently before his untimely demise in Kumbhakonam at the age of 32 due to tuberculosis.

In 1918, Indian mathematician Srinivas Ramanujan was admitted to the hospital in London, where he was visited by his colleague and long-time friend G H Hardy. The fellow mathematician had arrived in a taxi which was numbered '1729' and had thought about it on his way to the room, upon entering Ramanujan's room, Hardy blurted "it was rather a dull number," after a brief hello.

When Ramanujan came to know of the number, the mathematician said "No Hardy, it is a very interesting number. It is the smallest number

expressible as the sum of two cubes in two different ways." This conversation, which is the base of the mysterious Hardy-Ramanujan number is documented in his biography 'The man who knew infinity' by Robert Knaigel

The mystery of Ramanujan number

Ramanujan explained that 1729 is the only number that is the sum of cubes of two different pairs of numbers: **$12^3 + 1^3$, and $10^3 + 9^3$.**

His most popular discovery, however, remains the Hardy-Ramanujan number (also known as) to this date.

This certainly is a fascinating discovery and is the easiest to remember among all of Ramanujan's works.

This day was first observed when the then Prime Minister Dr Manmohan Singh declared Ramanujan's birthday as National Mathematics Day on 22 February 2012 upon his Madras University visit to pay tribute to Ramanujan's accomplishments and celebrated his 125th anniversary.

$$1^3 + 12^3 = 1729 = 9^3 + 10^3$$

1,729 is the smallest number which can be represented in two different ways as the sum of two cubes:

$$\begin{aligned} 1729 &= 1^3 + 12^3 \\ &= 9^3 + 10^3 \end{aligned}$$

It is also incidentally the product of 3 prime numbers:

$$1729 = 7 \times 13 \times 19$$

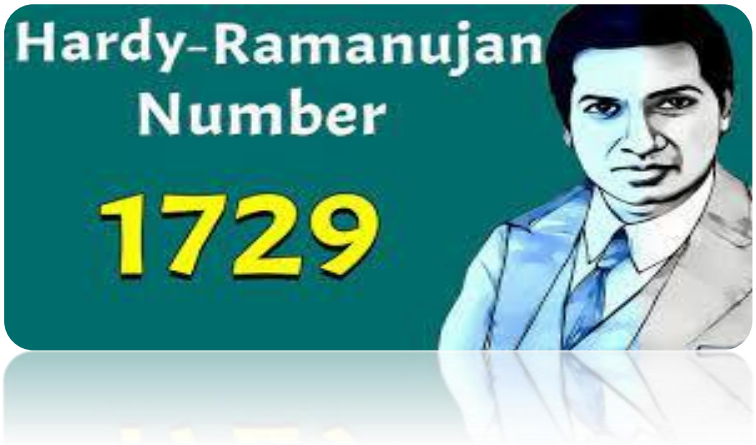
The largest known similar number is:

$$\begin{aligned} 885623890831 &= 7511^3 + 7730^3 \\ &= 8759^3 + 5978^3 \\ &= 3943 \times 14737 \times 15241 \end{aligned}$$

Ramanujan died aged 32 in Madras on April 26, 1920, due to hepatic amoebiasis caused by liver parasites. Hardy and Ramanujan's bond was firm. Hardy served as a father-figure to Ramanujan, a distant, impersonal father who was the ideal taskmaster and had high expectations of Ramanujan.

"*For my part, it is difficult for me to say what I owe to Ramanujan- his originality has been a constant source of suggestion to me ever since I knew him, and his death is one of the worst blows I have ever had.*"

–G. H. Hardy



10 MAJOR CONTRIBUTIONS AND ACHIEVEMENTS

Srinivasa Ramanujan FRS (1887 – 1920) was a self-taught Indian mathematical genius who made numerous contributions in several mathematical fields including mathematical analysis, infinite series, continued fractions, number theory and game theory. Ramanujan provided solutions to mathematical problems that were then considered unsolvable. Moreover, some of his work was so ahead of his time that mathematicians are still understanding its relevance. In 1914, Ramanujan found a formula for computing π (pi) that is currently the basis for the fastest algorithms used to calculate π. The circle method, which he developed with G. H. Hardy, constitute a large area of current mathematical research. Moreover, Ramanujan discovered K3 surfaces which play key roles today in string theory and quantum physics; while his mock modular forms are being used in an effort to unlock the secret of black holes. Know more about the achievements of Srinivasa Ramanujan through his 10 major contributions to mathematics.

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#1 HE WAS THE SECOND INDIAN TO BE ELECTED A FELLOW OF THE ROYAL SOCIETY

A self-taught genius, Ramanujan moved to England in *March 1914* after his talent was recognized by *British mathematician G. H. Hardy*. In 1916, Ramanujan was awarded a *Bachelor of Science by Research* degree (later named Ph.D.) by *Cambridge* even though he was not an undergraduate. The Ph.D. was awarded in recognition of his work on *'Highly composite numbers'*. In 1918, Ramanujan became *one of the youngest Fellows of the Royal Society* and only the *second Indian member*. The same year he was elected a *Fellow of Trinity College, Cambridge*, the *first Indian to be so honored*. During his short lifespan of 32 years, Ramanujan *independently compiled around 3,900 results*. Apart from the

below mentioned achievements his contributions include developing the relationship between *partial sums* and *hyper-geometric series*; independently discovering *Bernoulli numbers* and using these numbers to formulated *the value of Euler's constant up to 15 decimal places*; discovering the *Ramanujan prime number* and the *Landau–Ramanujan constant*; and coming up with *Ramanujan's sum* and the *Ramanujan's master theorem.*

#2 THE FASTEST ALGORITHMS FOR CALCULATION OF PI ARE BASED ON HIS SERIES

Finding an accurate approximation of π (pi) has been one of the most important challenges in the history of mathematics. In 1914, Srinivasa Ramanujan found a formula for computing pi that converges rapidly. His formula computes a further eight decimal places of π with each term in the series. It was in 1989, that Chudnovsky brothers computed π to over 1 billion decimal places on a supercomputer using a variation of Ramanujan's infinite series of π. This was a world record for computing the most digits of pi. Moreover, the Ramanujan series is currently the basis for the fastest algorithms used to calculate π.

For estimating π

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Ramanujan formula for estimating the value of Pi

#3 THE RAMANUJAN CONJECTURE PLAYED A KEY ROLE IN THE FAMOUS LANGLANDS PROGRAM

In 1916, Ramanujan published his paper titled "On certain arithmetical functions". In the paper, Ramanujan investigated the properties of Fourier coefficients of modular forms. Though the theory of modular forms was not even developed then, he came up with three fundamental conjectures that served as a guiding force for its development. His first two conjectures helped develop the Hecke theory, which was formulated 20 years after his paper, in 1936, by German mathematician Erich Hecke. However, it was his last conjecture, known as the Ramanujan conjecture, that created a sensation in in 20th century mathematics. It played a pivotal role in the Langlands program, which began in 1970 through the proposal of American-Canadian mathematician Robert Langlands. The Langlands program aims to relate representation theory and algebraic number theory, two seemingly different fields of mathematics. It is widely viewed as the single biggest project in modern mathematical research. "On certain arithmetical functions" by Ramanujan thus effectively changed the course of 20th century mathematics.

#4 HE DEVELOPED THE INFLUENTIAL CIRCLE METHOD IN PARTITION NUMBER THEORY

A partition for a positive integer n is the number of ways the integer can be expressed as a sum of positive integers. For example p(4) = 5. That means 4 can be expressed as a sum of positive integers in 5 ways: 4, 3+1, 2+2, 2+1+1 and 1+1+1 +1. Ramanujan, along with G. H. Hardy, invented the circle method which gave the first approximations of the partition of numbers beyond 200. This method was largely responsible for major advances in the 20th century of notoriously difficult problems such as Waring's conjecture and other additive questions. The circle method is now one of the central tools of analytic number theory. Moreover, circle method and its refinements constitute a large area of current mathematical research.

#5 HE DISCOVERED THE THREE RAMANUJAN'S CONGRUENCES

Related to the Partition Theory of Numbers, Ramanujan also came up with three remarkable congruences for the partition function p(n). They are p(5n+4) = 0(mod 5); p(7n+4) = 0(mod 7); p(11n+6) = 0(mod 11). For example, the first congruence means that if an integer is 4 more than a multiple of 5, then number of its partitions is a multiple of 5. The study of Ramanujan type congruence is a popular research topic of number theory. It was in 2011, that a conceptual explanation for Ramanujan's congruences was finally discovered. Ramanujan's work on partition theory has applications in a number of areas including particle physics (particularly quantum field theory) and probability.

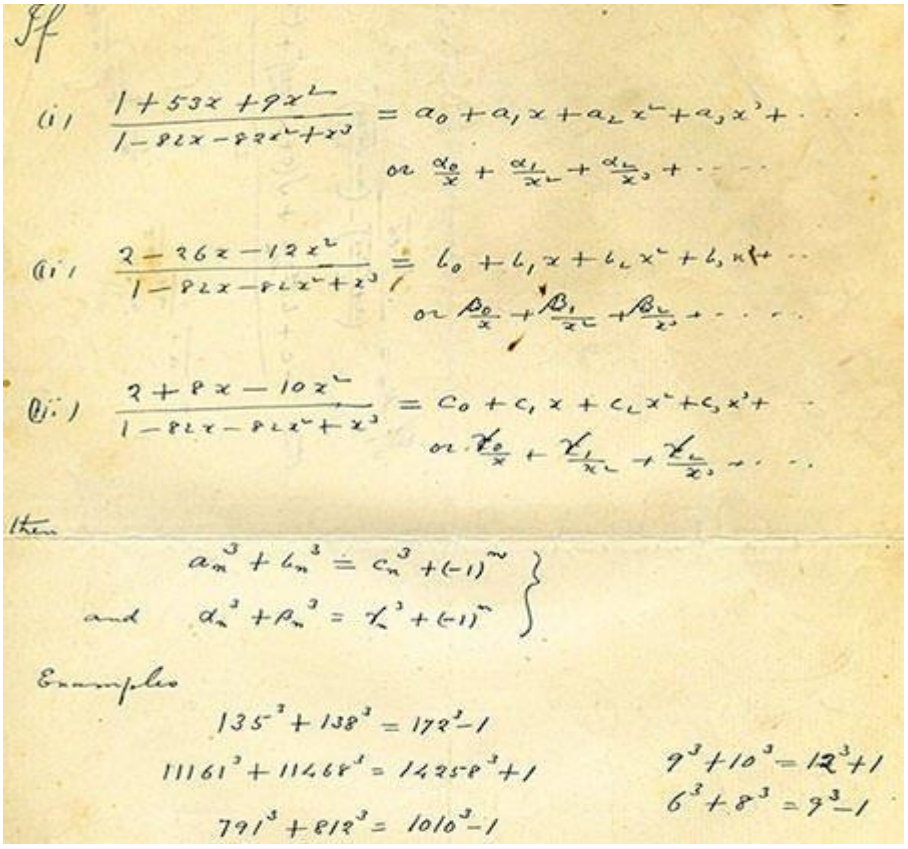
$$p(5k+4) = 0 \pmod 5$$
$$p(7k+5) = 0 \pmod 7$$
$$p(11k+6) = 0 \pmod{11}$$

#6 NUMBER 1729 IS NAMED HARDY-RAMANUJAN NUMBER

In a famous incident British mathematician G. H. Hardy while visiting Ramanujan had ridden in a taxi cab with the number 1729. He remarked to Ramanujan that the number "seemed to me rather a dull one, and that I hoped it was not an unfavourable omen". "No," Ramanajun replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." The two different ways are: $1729 = 1^3 + 12^3 = 9^3 + 10^3$. 1729 is now known as the Hardy–Ramanujan number. Moreover, numbers that are the smallest number that can be expressed as the sum of two cubes in n distinct ways are now referred to as taxicab numbers due to the incident. The relevance of 1729 has recently come to light as it was part of a much larger theory that Ramanujan was developing. Theorems have been established in theory of elliptic curves that involve this fascinating number.

#7 HE DID GROUNDBREAKING RESEARCH RELATED TO FERMAT'S LAST THEOREM

In 2013 famous Japanese American Mathematician Ken Ono, along with Sarah Trebat-Leder, found an equation by Ramanujan had clearly showed that he had been working on Fermat's last theorem, one of the most notable and difficult to prove theorems in the history of mathematics. In 1637, French mathematician Pierre de Fermat had asserted that: if n is a whole number greater than 2, then there are no positive whole number triples x, y and z, such that $x^n + y^n = z^n$. This means that there are no numbers which satisfy the equations: $x^3 + y^3 = z^3$; $x^4 + y^4 = z^4$; and so on. The equation of Ramanujan illustrates that he had found an infinite family of positive whole number triples x, y and z that very nearly, but not quite, satisfy Fermat's equation for n=3. They are off only by plus or minus one. Among them is 1729, which misses the mark by 1 for x=9, y=10 and z=12. Moving forward, Ramanujan also considered the equations of the form: $y^2 = x^3 + ax + b$. If you plot the points (x,y) for this equation you get an elliptic curve. Elliptic curves played a key role when English mathematician Sir Andrew Wiles finally proved Fermat's last theorem in 1994, a feat described as a "stunning advance" in maths.


Ff
(i) $\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$
or $\frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{x^3} + \cdots$
(ii) $\frac{2-36x-12x^2}{1-82x-82x^2+x^3} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$
or $\frac{\beta_0}{x} + \frac{\beta_1}{x^2} + \frac{\beta_2}{x^3} + \cdots$
(iii) $\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$
or $\frac{\gamma_0}{x} + \frac{\gamma_1}{x^2} + \frac{\gamma_2}{x^3} + \cdots$
then
$a_n^3 + b_n^3 = c_n^3 + (-1)^n$
and $\alpha_n^3 + \beta_n^3 = \gamma_n^3 + (-1)^n$
Examples
$135^3 + 138^3 = 172^3 - 1$
$11161^3 + 11468^3 = 14425^3 + 1$
$791^3 + 812^3 = 1010^3 - 1$
$9^3 + 10^3 = 12^3 + 1$
$6^3 + 8^3 = 9^3 - 1$

#8 RAMANUJAN WAS THE FIRST TO DISCOVER K3 SURFACES IN 1910S

Ken Ono also found that Ramanujan went on to discover an object more complicated than elliptic curves. When it was re-discovered in 1958 by Andre Weil, it was named K3 surface. Thus it has come to light that Ramanujan was using 1729 and elliptic curves to develop formulas for a K3 surface. "Elliptic curves and K3 surfaces form an important next frontier in mathematics and Ramanujan gave remarkable examples illustrating some of their features that we didn't know before." Moreover, K3 surfaces play key roles today in string theory and quantum physics. Like, string theory suggests that the world consists of more than the three dimensions that we can see. These extra dimensions are rolled up tightly in tiny little spaces too small for us to perceive. These tiny spaces have a particular geometric structure. Calabi–Yau manifold is a class of geometric objects that have similar structure and one of the simplest classes of Calabi-Yau manifolds comes from K3 surfaces.

#9 HIS THETA FUNCTION LIES AT THE HEART OF STRING THEORY IN PHYSICS

In mathematics, theta functions are special functions of several complex variables. German Mathematician Carl Gustav Jacob Jacobi came up with several closely related theta functions known as Jacobi theta functions. Theta functions were studied extensively by Ramanujan. He came up with the Ramanujan theta function, which generalizes the form of Jacobi theta functions while also capturing their general properties. In particular, the Jacobi triple product takes on an elegant form when written in terms of the Ramanujan theta function. Ramanujan theta function has several important applications. It is used to determine the critical dimensions in Bosonic string theory, superstring theory and M-theory.

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$$

#10 HIS MOCK MODULAR FORMS MAY UNLOCK THE SECRET OF BLACK HOLES

In a 1920 letter to Hardy, Ramanujan described several new functions that behaved differently from known theta functions, or modular forms, and yet closely mimicked them. These were the first ever examples of mock modular forms. More than 80 years later, in 2002, a description for these functions was provided by Sander Zwegers. Further, Ramanujan predicted that his mock modular forms corresponded to ordinary modular forms producing similar outputs for roots of 1. Ken Ono ultimately showed that a mock modular form could be computed just as Ramanujan predicted. It was found as the output of mock modular forms shoot off to enormous numbers, the corresponding ordinary modular form expand at a similar rate and thus their difference is a relatively small number. Expansion of mock modular forms is now used to compute the entropy, or level of disorder, of black holes. Thus even through black holes were virtually unknown during his time, Ramanujan was able to do maths which may unlock their secret.

Source: https://learnodo-newtonic.com/ramanujan-achievements

Mathematics Day

[A] National Mathematics Day [22 December]

National Mathematics Day is celebrated on 22 December annually to mark the birth anniversary of legendary Indian mathematician, Srinivasa Ramanujan and his contributions to the field of mathematics.

The main objective behind the celebration is to raise awareness among people about the importance of mathematics for the development of humanity. We can't ignore that several initiatives are taken to motivate, enthuse, and inculcate a positive attitude towards learning mathematics among the younger generation of the country. On this day, training is also provided to the mathematics teachers and students through camps and highlights the development, production, and dissemination of teaching-learning materials (TLM) for Mathematics and research in related areas.

In 2012, former Prime Minister Dr Manmohan Singh paid tribute to the mathematician during a ceremony in Chennai to commemorate Ramanujan's birth anniversary and declared December 22 as National Mathematics Day. Srinivasa Ramanujan was also seen on the 2012 India stamp. The Ramanujan Math Park located in Kuppam, of Andhra Pradesh, was inaugurated on this day in 2017.

National Mathematics Day is celebrated in various schools, colleges, universities, and educational institutions in India. Even the International Society of UNESCO (United Nations Educational, Scientific and Cultural Organisation) and India had agreed to work together to spread mathematics learning and understanding. Along with this, various steps were taken to educate the students in mathematics and spread knowledge to students and learners all over the world.

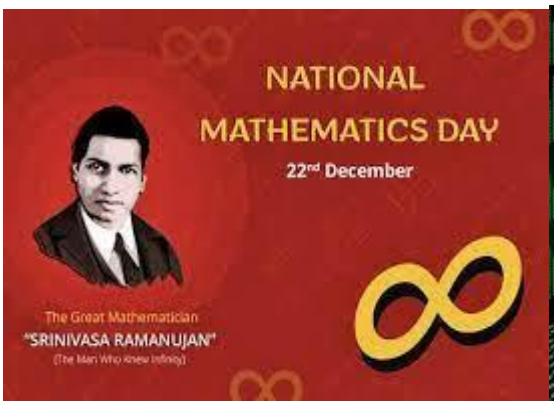
Mathematics Day is marked to raise awareness about the importance of mathematics. People celebrate this day to understand how maths is important for developing humanity from the grassroots level. On this day -

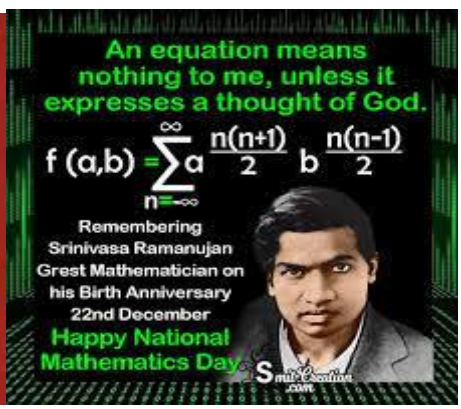
- The government takes several initiatives to enthuse, teach, and motivate the public and the country's youth to develop a positive attitude towards mathematics.
- Free training is provided to various students and mathematics teachers through camps. It highlights the production, dissemination, and development of mathematics teaching-learning materials.

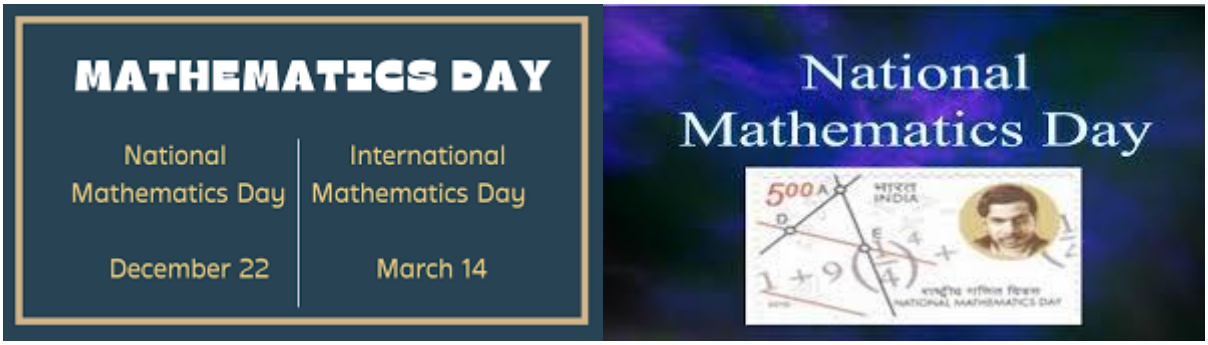
****

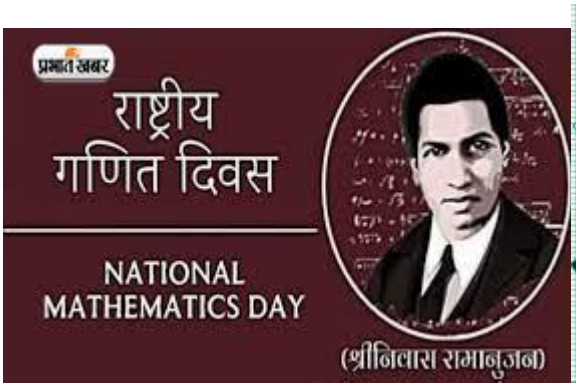
Prime Minister Dr. Manmohan Singh opens the 125th Birth Anniversary Celebrations of Mathematician Srinivas Ramanujan in Chennai.

India announced the birthday of celebrated mathematician Srinivas Ramanujan as 'National Mathematics Day'. The announcement was made by Prime Minister Dr. Manmohan Singh at the 125th Birth Anniversary celebrations of the mathematical genius in Chennai on Monday (December 26).

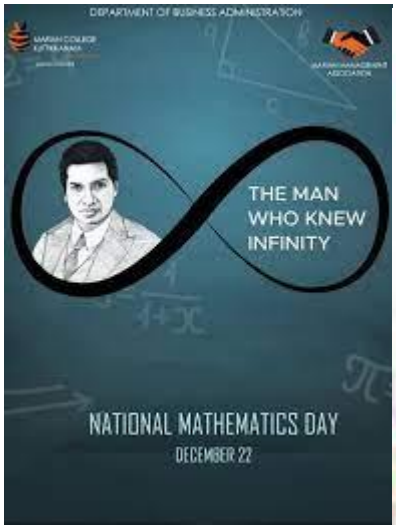


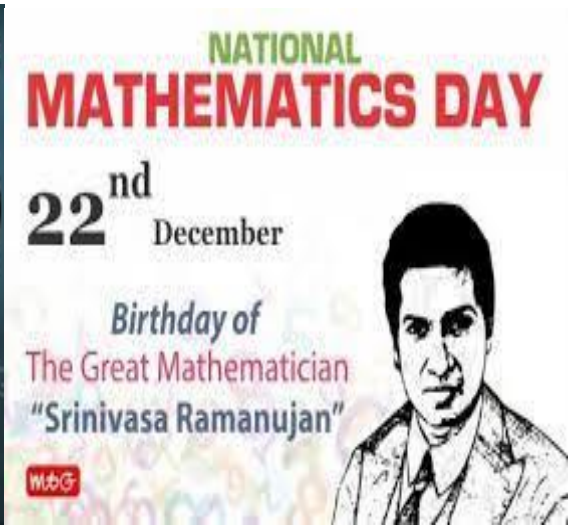


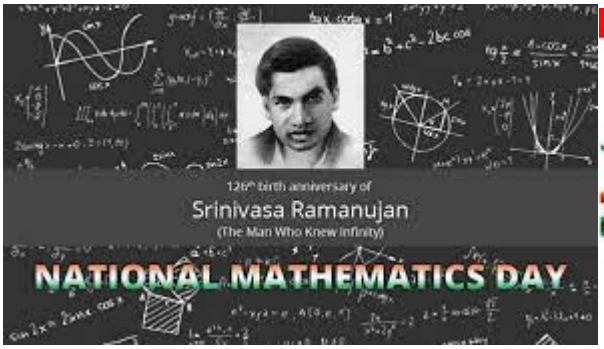

MATHEMATICS DAY
National Mathematics Day
December 22
International Mathematics Day
March 14
National Mathematics Day

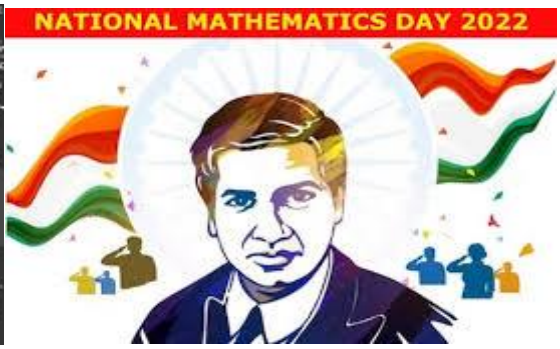

प्रभात खबर
राष्ट्रीय गणित दिवस
NATIONAL MATHEMATICS DAY
(श्रीनिवास रामानुजन)

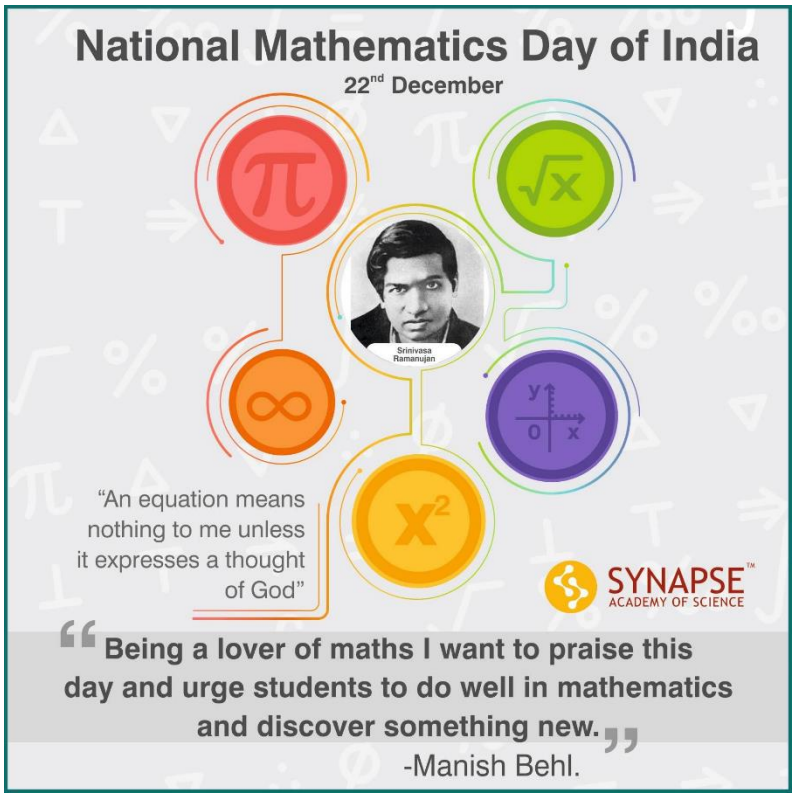
Math Day
Who discovered 0?
SRINIVASA RAMANUJAN


DEPARTMENT OF BUSINESS ADMINISTRATION
THE MAN WHO KNEW INFINITY
NATIONAL MATHEMATICS DAY
DECEMBER 22


NATIONAL
MATHEMATICS DAY
22nd December
Birthday of
The Great Mathematician
"Srinivasa Ramanujan"


126th birth anniversary of
Srinivasa Ramanujan
(The Man Who Knew Infinity)
NATIONAL MATHEMATICS DAY


NATIONAL MATHEMATICS DAY 2022


National Mathematics Day of India
22nd December
Srinivasa Ramanujan
"An equation means nothing to me unless it expresses a thought of God"
SYNAPSE
ACADEMY OF SCIENCE
"Being a lover of maths I want to praise this day and urge students to do well in mathematics and discover something new."
-Manish Behl.

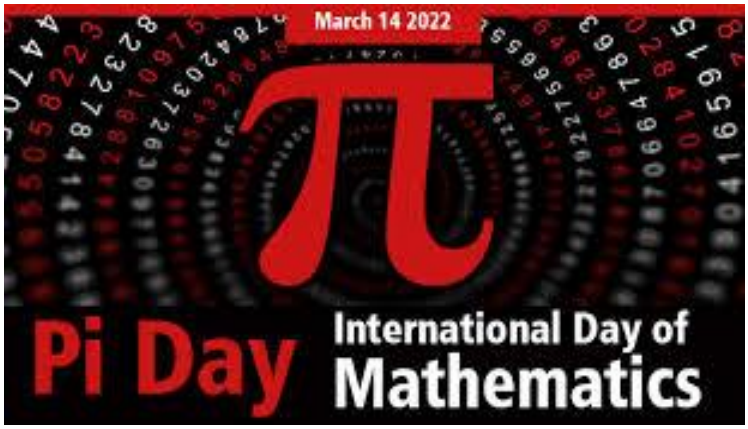
Some Selected Quotes:

- "Education is not just about going to school and getting a degree. It's about widening your knowledge and absorbing the truth about life." -Shakuntala Devi
- "Without mathematics, there's nothing you can do. Everything around you is mathematics. Everything around you is numbers." - Shakuntala Devi
- "Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding." -William Paul Thurston
- "Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore." -Albert Einstein
- "Mathematics is a game played according to certain simple rules with meaningless marks on paper." -David Hilbert
- "If there is a 50-50 chance that something can go wrong, then 9 times out of ten it will." -Paul Harvey

- "Mathematics is the most beautiful and most powerful creation of the human spirit." -Stefan Banach
- "Pure mathematics is, in its way, the poetry of logical ideas." -Albert Einstein
- "'Obvious' is the most dangerous word in mathematics." -Eric Temple Bell
- "The only way to learn mathematics is to do mathematics." -Paul Halmos
- "Nature is written in mathematical language." -Galileo Galilei

[B] International Day of Mathematics [14 March]

The International Day of Mathematics (IDM) is a worldwide celebration. Each year on March 14 all countries will be invited to participate through activities for both students and the general public in schools, museums, libraries and other spaces.



March 14 is already celebrated in many countries as Pi Day because that date is written as 3/14 in some countries and the mathematical constant Pi is approximately 3.14.

Every year we'll announce a new theme to flavor the celebration, spark creativity and bring light to connections between mathematics and all sorts of fields, concepts and ideas.

The IDM is supported by the following international and regional organizations:

- The International Council of Industrial and Applied Mathematics (ICIAM),
- The Committee for Women in Mathematics of IMU (CWM),
- The International Commission of Mathematical Instruction (ICMI),
- The African Mathematical Union (AMU),
- The European Mathematical Society (EMS),
- The Mathematical Council of the Americas (MCA)
- The Unión Matemática de América Latina y el Caribe (UMALCA),
- The Southeast Asian Mathematical Society (SEAMS)

- The Southern Africa Mathematical Sciences Association (SAMSA)
- The International Centre for Theoretical Physics (ICTP)
- The Centre international de mathématiques pures et appliquées (CIMPA)
- The Organization for Women in Science for the Developing World (OWSD)
- The African Women in Mathematics Association (AWMA)
- The African Mathematical Union Commission of African Women in Mathematics (AMU-CAWM)
- The European Women in Mathematics (EWM)
- International Science Council (ISC)
- Simons Foundation

HISTORY OF INTERNATIONAL DAY OF MATHEMATICS

The International Day of Mathematics is a global event that celebrates mathematics! It aims to educate people about the essential role of mathematics in science and technology, improve quality of life, empower women and girls, and contribute to sustainable development. This is a relatively new event that was created just a couple of years back.

The proclamation of March 14 as the International Day of Mathematics was adopted in the 205th session of the UNESCO's Executive Council. The day was then adopted in the 40th session of the General Conference of UNESCO in November 2019. Later in 2020, the world celebrated its First International Day of Mathematics on March 14, 2020, with the theme 'Mathematics is Everywhere.'

The major goals of the International Day of Mathematics is to improve understanding among the general public of the importance of mathematics in education, raise awareness of the role of maths in modern society, science, and disaster management; increase access to information about mathematics, increase international networking, and collaborations in public awareness of mathematics and more.

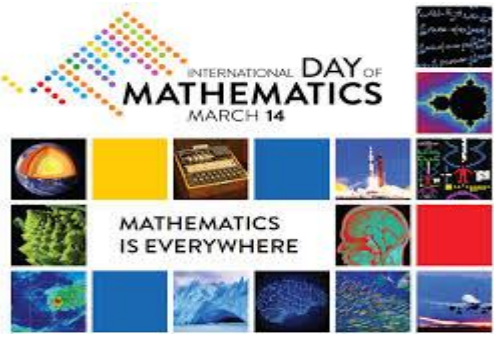
Each year, there is a new theme for the celebration of the International Day of Mathematics. Last year in 2021, the theme for this day was 'Mathematics for a better world.'

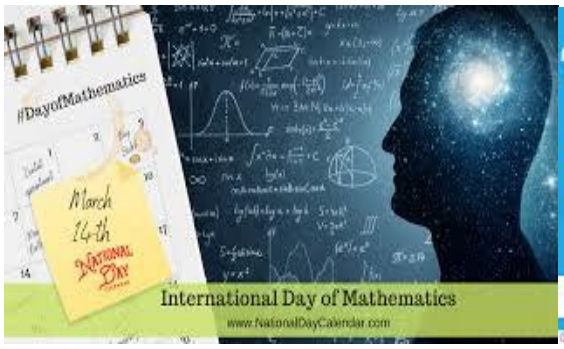
Mathematics is an interesting and engaging subject, and everyone needs to get accustomed to it. So, let's learn math, enjoy math, and celebrate math with all our hearts and brains!

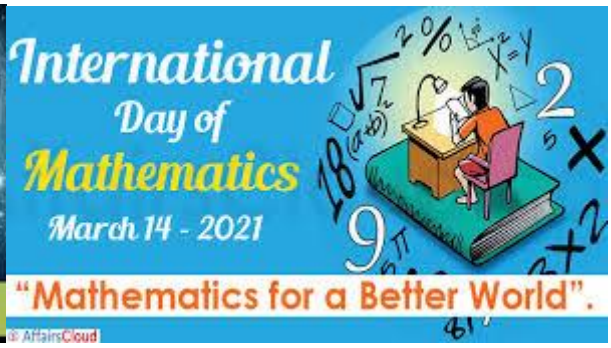
The theme for the **2023 International Day of Mathematics** is **Mathematics for Everyone**

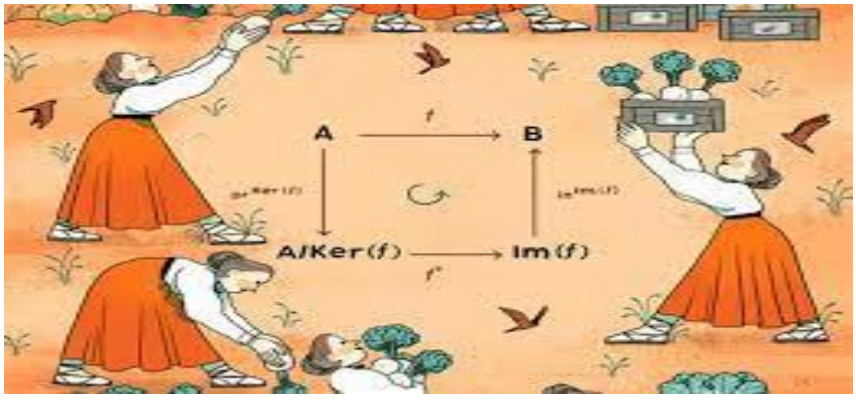
The theme was proposed by **Marco Zarco Rotairo** of the Trece Martires City National High School at Indang, Cavite, Philippines.

"I believe that Mathematics should be for everyone because all of us have mathematical ability, but only with varying extent and degree. Also, we must let everyone enjoy the wonders of Mathematics. The notion that Mathematics is only for the gifted and the genius must change."









Contribution to Mathematics

Srinivasa Ramanujan, the mathematical genius, came to be recognized only posthumously for his incredible contribution to the world of Mathematics. Leaving this world at the young age of 32, Srinivasa Ramanujan (1887-1920) contributed a great deal to mathematics that only a few could overtake in their lifetime.

Born in Erode (Tamil Nadu), Ramanujan demonstrated that he had an exceptional intuitive grasp of mathematics at a very young age. He began developing his theories in mathematics and published his first paper in 1911. In fact, he was the second Indian to be included as a Fellow of the Royal Society 9a fellowship of the world's most respected and famous scientists in 1918.

The field of number theory in mathematics was enriched with his intuitive research and his vast contribution. Every year, Srinivasa Ramanujan's birth anniversary on December 22 is commemorated as National Mathematics Day.

A wizard of intuition

Ramanujan has been recognized as one of the greatest mathematicians of his time. Surprisingly, he never had any formal training in mathematics. Most of his mathematics discoveries were based on sheer intuition, and most of them were proved to be right much later. GH Hardy, a famous British Mathematician, mentored him at Cambridge and encouraged Ramanujan to publish his findings in several papers.

Inspiring legacy

The Indian mathematician had few opportunities during his lifetime to showcase his talents. Still, his passion for giving his best to mathematics did not hold him back from leaving back his legacy for the world to marvel at. Ramanujan died at the age of 32 after contracting tuberculosis. But he has left behind a legacy that continues to inspire mathematicians to this day.

Major Contributions

- Ramanujan compiled around 3,900 results consisting of equations and identities. One of his most treasured findings was his infinite series for pi. This series forms the basis of many algorithms we use today. He gave several fascinating formulas to calculate the digits of pi in many unconventional ways.
- He discovered a long list of new ideas to solve many challenging mathematical problems, which gave a significant impetus to the development of game theory. His contribution to game theory is purely based on intuition and natural talent and remains unrivalled to this day.
- He elaborately described the mock theta function, which is a concept in the realm of modular form in mathematics. Considered an enigma till sometime back, it is now recognized as holomorphic parts of mass forms.
- One of Ramanujan's notebooks was discovered by George Andrews in 1976 in the library at Trinity College. Later the contents of this notebook were published as a book.
- 1729 is known as the Ramanujan number. It is the sum of the cubes of two numbers 10 and 9. For instance, 1729 results from adding 1000 (the cube of 10) and 729 (the cube of 9). This is the smallest number that can be expressed in two different ways as it is the sum of these two cubes. Interestingly, 1729 is a natural number following 1728 and preceding 1730.
- Ramanujan's contributions stretch across mathematics fields, including complex analysis, number theory, infinite series, and continued fractions.
- Ramanujan's other notable contributions include hypergeometric series, the Riemann series, the elliptic integrals, the theory of divergent series, and the functional equations of the zeta function.

Ramanujan left a number of unpublished notebooks filled with theorems that mathematicians have continued to study. G N Watson, Mason Professor of Pure Mathematics at Birmingham from 1918 to 1951 published 14 papers under the general title Theorems stated by Ramanujan and in all he published nearly 30 papers which were inspired by Ramanujan's work. Hardy passed on to Watson the large number of manuscripts of Ramanujan that he had, both written before 1914 and some written in Ramanujan's last year in India before his death.

Srinivasa Ramanujan's Contributions in Mathematics

Please visit this web link to read the following Articles

https://studiousguy.com/srinivasa-ramanujans-contributions-in-mathematics/

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- 1. Infinite series of π
- 2. Ramanujan number
- 3. Ramanujan Prime
- 4. Ramanujan Theta Function
- 5. Mock Theta Function
- 6. Partition
- 7. Ramanujan Magic Squares
- 8. Ramanujan Congruences
- 9. Highly composite numbers
- 10. Symmetric Equation by Ramanujan
- 11. Ramanujan-Nagell Equation
- 12. On Certain Arithmetical Functions
- 13. On Fermat's Last Theorem
- 14. Roger-Ramanujan Identities
- 15. Roger-Ramanujan Continued Fractions
- 16. Ramanujan's Master Theorem
- 17. Properties of Bernoulli Numbers
- 18. Euler Mascheroni Constant
- 19. Ramanujan Summation
- 20. Ramanujan Puzzles

Also visit this web link

https://www.iosrjournals.org/iosr-jm/papers/Vol12-issue3/Version-4/O120304137139.pdf

to read the Article:

"Srinivasa Ramanujan's Contributions in Mathematics"

Dharminder Singh1 , Arun Kumar Chopra2 , Sukhdeep Singh Bal3 1,2Asst. Professor Guru Nanak College, Budhlada, Dist. Mansa (Pb.), India 3 Student of B.Sc. III (N.M.) Guru Nanak College, Budhlada, Dist. Mansa (Pb.), India

Godfrey Harold Hardy

1878-1947: Professor of Pure Mathematics.

G.H. Hardy is primarily known for his achievements in number theory and mathematical analysis.

His essay of 1940 on the aesthetics of mathematics, *A Mathematician's Apology*, is considered one of the best insights into the mind of a working mathematician written for the layman.

G.H. Hardy's father was Bursar and Art Master at Cranleigh School; his mother had been a senior mistress at Lincoln Training College for teachers. Both parents were mathematically inclined. Hardy's own natural affinity for mathematics was perceptible at a young age. When just two years old, he wrote numbers up to millions, and when taken to church he amused himself by factorizing the numbers of the hymns.

After schooling at Cranleigh, Hardy was awarded a scholarship to Winchester College for his mathematical work. In 1896 he entered Trinity, and after only two years of preparation he came fourth in the Mathematics Tripos. Years later, Hardy sought to abolish the Tripos system, as he felt that it was becoming more an end in itself than a means to an end. While at university, Hardy joined the Cambridge Apostles.

As his most important influence Hardy cited the self-study of *Cours d'analyse de l'École Polytechnique* by the French mathematician Camille Jordan, through which he became acquainted with the more precise mathematics tradition in continental Europe. In 1900 he passed part II of the tripos and was awarded a fellowship. In 1903 he earned his M.A., which was the highest academic degree at English universities at that time. From 1906 onward he held the position of a lecturer where teaching six hours a week left him time for research.

In 1919 he left Cambridge to take the Savilian Chair of Geometry at Oxford in the aftermath of the Bertrand Russell affair during World War I. He returned to Cambridge in 1931, where he was Sadleirian Professor until 1942.

Hardy is credited with reforming British mathematics by bringing rigour into it, which was previously a characteristic of French, Swiss and German mathematics. British mathematicians had remained largely in the tradition of applied mathematics, in thrall to the reputation of Isaac Newton (see Cambridge Mathematical Tripos). Hardy was more in tune with the *cours d'analyse* methods dominant in France, and aggressively promoted his conception of pure mathematics, in particular against the hydrodynamics which was an important part of Cambridge mathematics.

From 1911 he collaborated with J.E. Littlewood, in extensive work in mathematical analysis and analytic number theory. This (along with much else) led to quantitative progress on the Waring problem, as part of the Hardy-Littlewood circle method, as it became known. In prime number theory, they proved results and some notable conditional results. This was a major factor in the development of number theory as a system of conjectures; examples are the first and second Hardy-Littlewood conjectures. Hardy's collaboration with Littlewood is among the most successful and famous collaborations in mathematical history. In a 1947 lecture, the Danish mathematician Harald Bohr reported a colleague as saying,

"Nowadays, there are only three really great English mathematicians: Hardy, Littlewood, and Hardy-Littlewood."
Starting in 1914, he was the mentor of the Indian mathematician Srinivasa Ramanujan, a relationship that has become celebrated. Hardy almost immediately recognized Ramanujan's extraordinary albeit untutored brilliance, and Hardy and Ramanujan became close collaborators. In an interview by Paul Erdős, when Hardy was asked what his greatest contribution to mathematics was, Hardy unhesitatingly replied that it was the discovery of Ramanujan. He called their collaboration "the one romantic incident in my life."

Hardy is also known for formulating the Hardy-Weinberg principle, a basic principle of population genetics, independently from Wilhelm Weinberg in 1908. He played cricket with the geneticist Reginald Punnett who introduced the problem to him, and Hardy thus became the somewhat unwitting founder of a branch of applied mathematics. He preferred his work to be considered *pure mathematics*, perhaps because of his detestation of war and the military uses to which mathematics had been applied, stating: *"I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."*

His collected papers were published by OUP. *The Indian Clerk* (2007) is a novel by David Leavitt based on Hardy's life at Cambridge, including his discovery of and relationship with Srinivasa Ramanujan.

Resource:

http://trinitycollegechapel.com/about/memorials/brasses/hardy/

The Woman Who Transcended Infinity

*Great men often had selfless partners resolutely supporting them in pursuit of their dreams. This is the story of one such woman, **Janaki Ammal**, wife of the celebrated mathematician, Srinivasa Ramanujan.*

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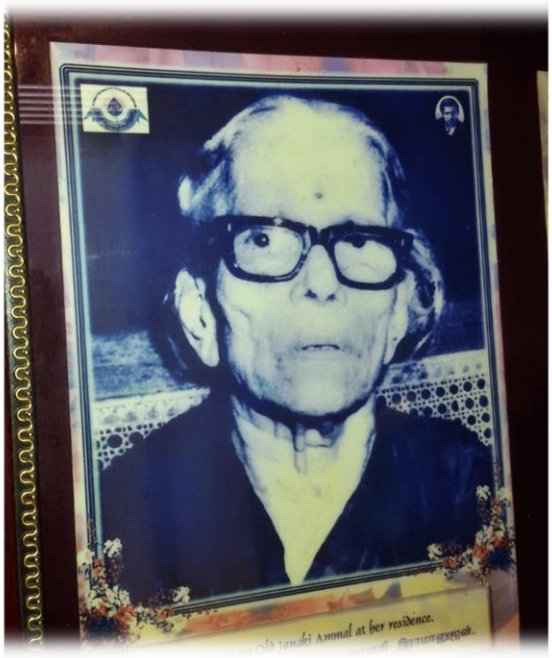

Janaki Ammal at her residence.

Janaki Ammal

wife of the celebrated mathematician Srinivasa Ramanujan

December 22nd is the birth anniversary of Srinivasa Ramanujan, the mathematics genius. Most of us have heard of Ramanujan. Thanks to the film 'The man who knew infinity', his life story is quite well known. But how many of us know about the woman who stood by him during his troubled years — his courageous wife who continued to celebrate him in her own quiet resilient way all her life? This story is about her — the woman who knew Infinity and perhaps transcended it too.

Her name was Janaki Ammal. She was barely 10 when she was married to the 21-year-old Ramanujan in 1909. Since she was too young to run a house, Janaki lived with her parents for the next 3 years. In 1912, Ramanujan got a job in the Madras (Chennai) Port Trust and Janaki's parents thought she was ready for the life of a householder. The couple moved into a small apartment in Triplicane, along with Ramanujan's mother, Komalatammal. Was this the beginning of a romantic phase in her life? Hardly.

For one, they were constantly under the watchful eye of her mother-in-law. Remember, Janaki was not even 14. For another, even by the conservative standards of those times, Ramanujan must have been an unromantic husband. She observed that he spent most of his free time in mathematical research and not even a loving wife could distract him from this first love. Despite all this, Janaki adjusted quickly to her new life. Perhaps romance would bloom in good time?

In 1914, Ramanujan had a breakthrough. He was invited to study mathematics in Cambridge. This was an opportunity of a lifetime but Ramanujan was in a dilemma! Overseas travel was taboo to orthodox Hindus, because they believed it would pollute the soul. Anybody who violated the taboo lost his caste and social status and was even ostracised.

Ramanujan was a deeply religious man and this complex problem tortured him intensely. He finally resolved it the religious way. After praying to his family deity, Namagiri Thaayaar, he declared that she had blessed the overseas trip. Janaki, who hadn't spent any time alone with Ramanujan in their brief time together, hoped to travel with him. By then, Janaki was 15 years old and naively asked Ramanujan if she could travel with him. However, Ramanujan declined, saying she was probably too young to travel. He was to regret that decision later.

Leaving behind his wife and mother, Ramanujan left for Cambridge. At Cambridge, he immersed himself in the joys of research . Those were his brightest years. Those were also his darkest years. He was alone in an alien culture. He was socially inept, with no friends except his own mentors in the mathematics department. The taboo still rankled in his mind. He was wary of eating outside, because even vegetarian food probably did not conform to puritanical Hindu standards. Good food was scarce anyway during war time. He cooked his own food at odd times during occasional breaks from research. He was overworked and undernourished, so something had to give: his body. At the end of 5 years, he had earned fellowships at the Trinity College of Cambridge and the Royal Society, but lost his health. Reluctantly, he decided to return to Madras in 1919. In all this time, Ramanujan wrote letters to Janaki, but she never got to see most of them. For reasons best known to her, Ramanujan's mother hid these letters from Janaki. She even hid his letter announcing the date of his return to India, so Janaki did not go to receive him.

Once home, he continued to nurture ambitions of returning to England after he recovered. He told Janaki that he would buy her diamond earrings and that she did not have to live in poverty ever again. During this time they got very little help from their own relatives because no one wanted to associate with a man who had broken the taboo. Janaki continued to tenderly nurse Ramanujam, hoping he would turn around. Ramanujan was overwhelmed and told her that if she had only accompanied him to England, things would not have become so dire.

Deep inside, Ramanujan knew that his life was rapidly fading away. He feverishly worked on his last theorems from his bed. Janaki did not complain because at last, she had Ramanujan all to herself. But it was too good to last. Within a year Ramanujan died. His relatives boycotted the funeral and did not support Janaki.

But this determined woman was not ready to give up. Janaki, barely 22, went to live with her brother in Bombay (Mumbai). There she educated herself in English and tailoring. Not wanting to be a burden, she returned to Madras and settled in Triplicane – the same neighbourhood where she and Ramanujan had started life together. She opened a tailoring school and shop to supplement the small pension she received from Madras University. Though the income was small, she was in charge of her own life . She had the respect and support of the small local community. Then came a tragedy which became a life defining moment.

One of her close friends in Triplicane was a single mother named Soundaravalli. She died suddenly in 1950, leaving an orphan son, Narayanan, who was just 7 years old. Janaki made a bold decision: she adopted Narayanan. Even though she was already 50 years old, she decided to take the plunge. She educated Narayanan until he got a job in a reputed Bank. In 1972 she conducted his wedding with Vaidehi, a girl from the same bank.

Unlike in her own marriage, she encouraged her son and daughter-in-law to move into a house nearby. She loved them but did not want to stifle their independence. She knew from her own experience that youngsters needed space! Janaki continued her independent life, living frugally and spending her savings by paying school fees for poor students. Some children would ask her for a small part of their examination fees, because her blessing was considered auspicious! When she was nearly 90, she retired

and moved under the loving care of the grateful Narayanans. She had always been a giver who found joy in giving. Now it was her turn to receive.

She passed away peacefully in 1994, aged 95 years. She was just 5 years short of 100. But 100 was a mere number.

Janaki had transcended infinity!

Watch the "Untold Story of Ramanujan's Wife"

https://www.youtube.com/watch?v=xkKJ6qYgS5s&ab_channel=Mathopia

Janakiammal Ramanujan 1987 Channel 4 "Equinox" Interview

Visit to watch 'The Interview'

https://www.youtube.com/watch?v=1qaRtMjTD-k&ab_channel=Kasturi

Ramanujan with his Parents!

PAGE on Srinivasa Ramanujan in
The Institute of Mathematical Sciences, Madras.

Srinivasa Ramanujan

Born: 22 Dec 1887 in Erode, Tamil Nadu state, India
Died: 26 April 1920 in Madras, Tamil Nadu state, India

Ramanujan was one of India's greatest mathematical geniuses. He made substantial contributions to the analytical theory of numbers and worked on elliptic functions, continued fractions, and infinite series.

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SRINIVASA RAMANUJAN

(Dec. 22, 1887 -- April 26, 1920)

K. Srinivasa Rao

The Institute of Mathematical Sciences, Madras-600 113.

Srinivasa Ramanujan (1887-1920) hailed as an all-time great mathematician, like Euler, Gauss or Jacobi, for his *natural genius*, has left behind 4000 original theorems, despite his lack of formal education and a short life-span. In his formative years, after having failed in his F.A. (First examination in Arts) class at College, he ran from pillar to post in search of a benefactor. It is during this period, 1903-1914, he kept a record of the final results of his original research work in the form of entries in two large-sized Note Books. These were the ones which he showed to Dewan Bahadur Ramachandra Rao (Collector of Nellore), V. Ramaswamy Iyer

(Founder of Indian Mathematical Society), R. Narayana Iyer (Treasurer of IMS and Manager, Madras Port Trust), and to several others to convince them of his abilities as a Mathematician. The orchestrated efforts of his admirers, culminated in the encouragement he received from Prof. G.H. Hardy of Trinity College, Cambridge, whose warm response to the historic letter of Ramanujan which contained about 100 theorems, resulted in inducing the Madras University, to its lasting credit, to rise to the occasion thrice - in offering him the first research scholarship of the University in May 1913 ; then in offering him a scholarship of 250 pounds a year for five years with 100 pounds for passage by ship and for initial outfit to go to England in 1914 ; and finally, by granting Ramanujan 250 pounds a year as an allowance for 5 years commencing from April 1919 soon after his triumphant return from Cambridge ``with a scientific standing and reputation such as no Indian has enjoyed before".

Ramanujan was awarded in 1916 the B.A. Degree by research of the Cambridge University. He was elected a Fellow of the Royal Society of London in Feb. 1918 being a ``Research student in Mathematics Distinguished as a pure mathematician particularly for his investigations in elliptic functions and the theory of numbers" and he was elected to a Trinity College Fellowship, in Oct. 1918 (- a prize fellowship worth 250 pounds a year for six years with no duties or condition, which he was not destined to avail of). The ``Collected Papers of Ramanujan" was edited by Profs. G.H.Hardy, P.V. Seshu Aiyar and B.M. Wilson and first published by Cambridge University Press in 1927 (later by Chelsea, 1962 ; and by Narosa, 1987), seven years after his death. His `Lost' Notebook found in the estate of Prof. G.N. Watson in the spring of 1976 by Prof. George Andrews of Pennsylvania State University, and its facsimile edition was brought out by Narosa Publishing House in 1987, on the occasion of Ramanujan's birth centenary. His bust was commissioned by Professors R. Askey, S. Chandrasekhar, G.E. Andrews, Bruce C. Berndt (`the gang of four'!) and `more than one hundred mathematicians and scientists who contributed money for the bust' sculpted by Paul Granlund in 1984 and another was commissioned for the Ramanujan Institute of the University of Madras, by Mr. Masilamani in 1994. His original Note Books have been edited in a series of five volumes by Bruce C. Berndt (``Ramanujan Note

Books", Springer, Parts I to V, 1985 onwards), who devoted his attention to each and every one of the three to four thousand theorems. Robert Kanigel recently wrote a delightfully readable biography entitled : ``The Man who knew Infinity : a life of the Genius Ramanujan" (Scribners 1991; Rupa & Co. 1993). Truly, the life of Ramanujan in the words of C.P. Snow: ``is an admirable story and one which showers credit on nearly everyone".

During his five year stay in Cambridge, which unfortunately overlapped with the first World War years, he published 21 papers, five of which were in collaboration with Prof. G.H. Hardy and these as well as his earlier publications before he set sail to England are all contained in the ``Collected Papers of Srinivasa Ramanujan", referred earlier. It is important to note that though Ramanujan took his ``Note Books" with him he had no time to delve deep into them. The 600 formulae he jotted down on loose sheets of paper during the one year he was in India, after his meritorious stay at Cambridge, are the contents of the `Lost' Note Book found by Andrews in 1976. He was ailing throughout that one year after his return from England (March 1919 - April 26, 1920). The last and only letter he wrote to Hardy, from India, after his return, in Jan. 1920, four months before his demise, contained no news about his declining health but only information about his latest work : ``I discovered very interesting functions recently which I call `Mock' theta-functions. Unlike the `False' theta-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as ordinary theta-functions. I am sending you with this letter some examples ... ". The following observation of Richard Askey is noteworthy: ``Try to imagine the quality of Ramanujan's mind, one which drove him to work unceasingly while deathly ill, and one great enough to grow deeper while his body became weaker. I stand in awe of his accomplishments; understanding is beyond me. We would admire any mathematician whose life's work was half of what Ramanujan found in the last year of his life while he was dying".

As for his place in the world of Mathematics, we quote Bruce C Berndt: ``Paul Erdos has passed on to us Hardy's personal ratings of

mathematicians. Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100, Hardy gave himself a score of 25, Littlewood 30, Hilbert 80 and Ramanujan 100". G.H.Hardy, in 1923, edited Chapter XII of Ramanujan's second Notebook on Hypergeometric series which contained 47 main theorems, many of them followed by a number of corollaries and particular cases. This work had taken him so many weeks that he felt that if he were to edit the entire Notebooks ``it will take the whole of my lifetime. I cannot do my own work. This would not be proper." He urged Indian authorities and G.N.Watson and B.M. Wilson to edit the Notebooks. Watson and Wilson divided the task of editing the Notebooks - Chapters 2 to 13 were to be edited by Wilson and Chapters 14 to 21 by Watson. Unfortunately, the premature death of Wilson, in 1935, at the age of 38, aborted this effort. In 1957, with monetary assistance from Sir Dadabai Naoroji Trust, at the instance of Professors Homi J Bhabha and K. Chandrasekaran, the Tata institute of Fundamental Research published a facsimile edition of the Notebooks of Ramanujan in two volumes, with just an introductory para about them. The formidable task of truly editing the Notebooks was taken up in right earnest by Professor Bruce C. Berndt of the University of Illinois, in May 1977 and his dedicated efforts for nearly two decades has resulted in the *Ramanujan's Notebooks* published by Springer-Verlag in five Parts, the first of which appeared in 1985. The three original Ramanujan Notebooks are with the Library of the University of Madras, some of the correspondence, papers/letters on or about Ramanujan are with the National Archives at New Delhi and the Tamil Nadu Archives, and a large number of his letters and connected papers/correspondence and notes by Hardy, Watson, Wilson are with the Wren Library of Trinity College, Cambridge. ``Ramanujan : Letters and Commentary", by Bruce C. Berndt and Robert A. Rankin (published jointly by the American Mathematical Society and London Math. Society, 1995) is a recent publication. The Ramanujan Institute for Advanced Study in Mathematics of the University of Madras is situated at a short distance from the famed Marina Beach and is close to the Administrative Buildings of the University and its Library. The bust of Ramanujan made by Mr. Masilamani is housed in the Ramanujan Institute. In 1992, the Ramanujan Museum was started in the Avvai Kalai

Kazhagam in Royapuram. Mrs. Janakiammal Ramanujan, the widow of Ramanujan, lived for several decades in Triplicane, close to the University's Marina Campus and died on April 13, 1994. A bust of Ramanujan, sculpted by Paul Granlund was presented to her and it is now with her adopted son Mr. W. Narayanan, living in Triplicane.

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Ramanujan's Birthday Magic Square

One of the world's most talented mathematicians, Srinivasa Ramanujan (1887–1920), was born on 22nd December 1887. Some of Ramanujan's work on recreational mathematics and magic squares can be found in his early notebooks. A magic square is an NxN matrix in which every row, column, and diagonal sums up to the same number. The sum is called the magic constant or magic sum of the magic square. Ramanujan created the following birthday magic square from his date of birth (in DD MM YYYY format) where all four rows, four columns and two diagonals sum up to 139.

$$\begin{bmatrix} 22 & 12 & 18 & 87 \\ 88 & 17 & 09 & 25 \\ 10 & 24 & 89 & 16 \\ 19 & 86 & 23 & 11 \end{bmatrix}$$

Ramanujan's Birthday Magic Square

e.g. sum of **all rows** equals to 139.
22 + 12 + 18 + 87 = 139; 88 + 17+ 09 + 25 = 139; 10 + 24 + 89 + 16 = 139; 19 + 86 + 23 + 11 = 139

sum of **all columns** equals to 139
22 + 88 + 10 + 19 = 139; 12 + 17 + 24 + 86 = 139; 8 + 09 + 89 + 23 = 139; 87 + 25 + 16 + 11 = 139

sum of **two diagonals** equals to 139 as well

22 + 17 + 89 + 11 = 139; 19 + 24 + 09 + 87 = 139

In addition, the **2x2 matrices in all four corners** also sum up to 139
22 + 12 + 88 + 17 = 139; 18 + 87 + 09 + 25 =139;
10 + 24 + 19 + 86 = 139; 89 + 16 + 23+ 11=139;

sum of all the corner digits and **central 2x2 matrix** is also 139.
22 + 87 + 19 + 11 =139; 17 + 09 + 24 + 89 =139;

That seems like magic and hence the name — magic square. As depicted in the figure 1, there are different types of magic squares. Ramanujan's birthday magic square is a complete magic square. Magic squares are not purely recreational, it has applications in Birkhoff — von Neumann decomposition, Quantum permutation matrices etc.

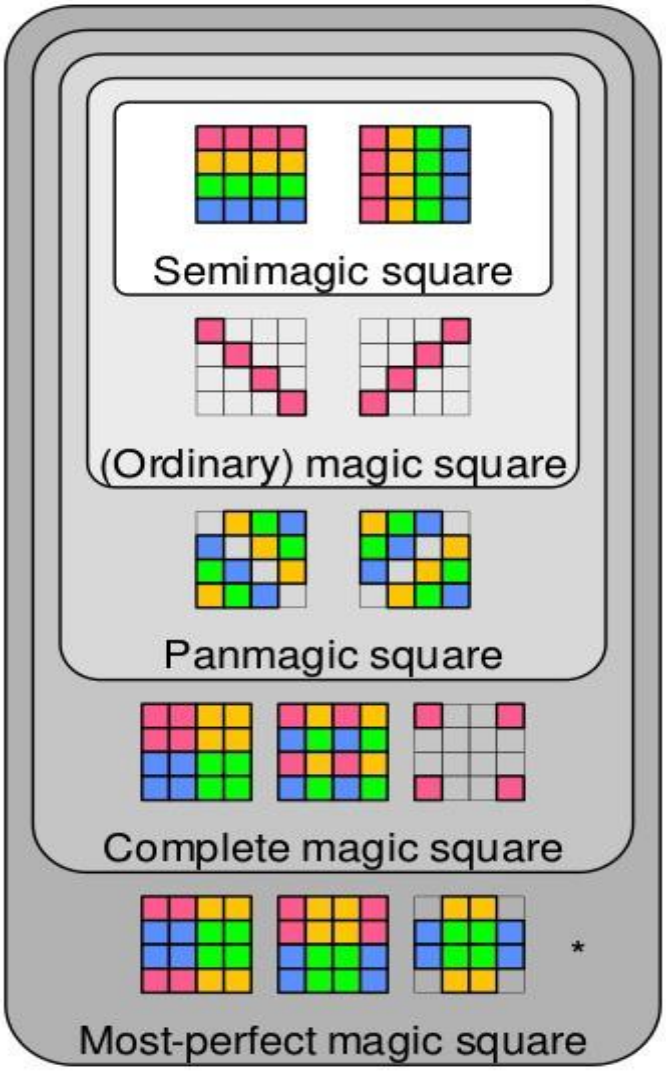

Figure -1 Euler diagram of requirements of some types of 4 x 4 magic squares. Cells of the same color sum to the magic constant.

But how do we construct birthday magic square(s)? If we inspect the matrix closely, we will notice there is a structure in it. If we denote:

a = two digits date, b = two digits month, c = first two digits of year, d = last two digits of the year, in other words a=22, b=12, c=18, d=87.

the structure is …

$$\begin{bmatrix} a & b & c & d \\ (d+1) & (c-1) & (b-3) & (a+3) \\ (b-2) & (a+2) & (d+2) & (c-2) \\ (c+1) & (d-1) & (a+1) & (b-1) \end{bmatrix}$$

****

You can create a "Magic Square" from your date of birth.

Watch the following Video

Magic Square from your DATE OF BIRTH
Ramanujan's Magic

https://www.youtube.com/watch?v=XAGUodW48fE&ab_channel=SureshAggarwal

Try and enjoy!

BIRTHDAY MAGIC SQUARE

We are introducing on Srinivasa Ramanujan Birthday through magic square in the class room at high school level to know better, one of the contributions of Ramanujan. It will enrich mathematics learning and ensure developing interest among students. By learning magic square the students may commemorate Birthday of Ramanujan, falls on 22nd December 1887.

Formula 1 , for the formulation of magic square

A	B	C	D
D+1	C-1	B-3	A+3
B-2	A+2	D+2	C-2
C+1	D-1	A+1	B-1

Putting A=22, B=12, C=18 and D=87 in formula 1, we get the greatest mathematician BIRTHDAY MAGIC SQUARE

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

From the table, we can get the so many properties

(i) Sum of the numbers of any column is 139

$$C1 \Rightarrow A + D + 1 + B - 2 + C + 1 = A + B + C + D$$
$$C2 \Rightarrow B + C - 1 + A + 2 + D - 1 = A + B + C + D$$
$$C3 \Rightarrow C + B - 3 + D + 2 + A + 1 = A + B + C + D$$
$$C4 \Rightarrow D + A + 3 + C - 2 + B - 1 = A + B + C + D$$

(ii) Sum of the numbers of any row is 139

$$R1 \Rightarrow A + B + C + D$$
$$R1 \Rightarrow D + 1 + C - 1 + B - 3 + A + 3 = A + B + C + D$$
$$R3 \Rightarrow B - 2 + A + 2 + D + 2 + C - 2 = A + B + C + D$$
$$R4 \Rightarrow C + 1 + D - 1 + A + 1 + B - 1 = A + B + C + D$$

(iii) Sum of numbers of any diagonals is 139

$$D1 = A + C - 1 + D + 2 + B - 1 = A + B + C + D$$
$$D2 = C + 1 + A + 2 + B - 3 + D = A + B + C + D$$

(iv) Sum of the corner number is 139

$$A + B + C + 1 + B - 1 = A + B + C + D$$

(v) Sum of the corner square is 139

$A + B + D + \mathbf{1} + C - \mathbf{1} = A + B + C + D$
$C + D + B - \mathbf{3} + A + \mathbf{3} = A + B + C + D$
$B - \mathbf{2} + A + \mathbf{2} + C + \mathbf{1} + D - \mathbf{1} = A + B + C + D$
$D + \mathbf{2} + C - \mathbf{2} + A + \mathbf{1} + B - \mathbf{1} = A + B + C + D$

(vi) Sum of the middle square is 139

$C - \mathbf{1} + B - \mathbf{3} + A + \mathbf{2} + D + \mathbf{2} = A + B + C + \mathrm{D}$

Formula (2) similarly one can find another type of formula table

A	B	C	D
C+3	D-3	A+3	B-3
D-2	C-2	B+2	A+2
B-1	A+5	D-5	C+1

Solution:

22	12	18	87
21	84	25	09
85	11	14	24
11	27	82	19

Formula (3)

A	B	C	D
C-1	D+1	A-1	B+1
D+1	C+1	B-1	A-1
B	A-2	D+2	C

Solution:

22	12	18	87
17	88	21	13
88	19	11	21
12	20	89	18

This experience will be developing creative skills ,as one can find on their own different types of formula and compute it to form his own birthday magic square. This interest will let students to introduce Birthday magic square to his friends/brothers/sisters/teachers.This was fitted gift by Ramanujan to the students of nation.

Author: K.Thirumurugan

He has been actively engaged in the school maths, joyful teaching of math, discovering innovative idea, served on resource person, published several papers and articles.

A PILGRIMAGE TO RAMANUJAN'S HOMETOWN

Krishnaswami Alladi
Department of Mathematics
University of Florida
Gainesville, FL 32611

The district of Tanjore (*Tanjavur* in Tamil) in the state of Tamil Nadu in South India, has been a seat of culture for several centuries. Tanjore has produced some of the greatest composers of *Carnatic* music, the classical music of South India. Tanjore is also very well known for art in various forms; in particular, Tanjore paintings of Hindu gods which have crystal glass pieces imbedded in them, are much appreciated both for their special beauty and art value. The Tanjore district also has the greatest concentration of Hindu temples, and some of them are architectural marvels. It is in the midst of this region steeped in culture, that the Indian mathematical genius Srinivasa Ramanujan lived in the town of *Kumbakonam* in Tanjore district and made some of his path breaking discoveries. In December 2003, my wife Mathura and I had an opportunity to visit Kumbakonam, see Ramanujan's humble home from which a thousand theorems emerged, and visit several temples in that area including the one next to Ramanujan's home where he prayed regularly. In this article I shall describe our memorable visit to Kumbakonam and Tanjore. But to prepare a suitable background, I first will briefly describe Ramanujan's fascinating life story.

Ramanujan: Srinivasa Ramanujan, one of the greatest mathematicians in history, belonged to an orthodox Hindu brahmin family in the town of Kumbakonam. He was born in Erode in South India on December 22, 1887. Erode was where his mother's family lived. It is the Hindu custom that even though after marriage a woman would live in her husband's home, she will go to back to her family home to give birth. Among the

brahmins, there are two main subcastes in South India, the *Iyers* who worship Lord Shiva the Destroyer as the primary diety, and the *Iyengars*, who worship Lord Vishnu the Protector as the main diety. Ramanujan was an Iyengar, and from his parents he learnt many verses of the *Vedas*, the Hindu holy scriptures, as well as stories from Hindu folklore and epics. Ramanujan and his family offered prayers regularly at the *Sarangapani temple* for Lord Vishnu in Kumbakonam, which was just down the street from his home.

Ramanujan showed his unsual talent for mathematics very early. Often in the middle of the night he would get up and write down mathematical formulae on a piece of slate lest he should forget them in the morning when he woke up. He would then record these marvelous formulae in his now famous notebooks. Ramanujan had a special veneration for the Goddess *Namagiri* of the temple in the neighbouring town of *Namakkal*, and we are told that the Goddess of Namakkal would come in his dreams and give him these formulae! In the town high school in Kumbakonam, Ramanujan's teachers realized that he was unusually talented, but they obviously could not understand or judge the importance of his discoveries. Ramanujan later moved to *Madras* (now called Chennai), the capital and largest city in Tamil Nadu, where he attended College. Although he was successful in high school, his obsessive preoccupation with mathematics led to a neglect of other subjects and so he had to drop out of college. The advantage of being in Madras was that he could come in contact with persons, both Englishmen and Indians, who could appreciate his work. Some of them suggested that he should communicate his findings to leading mathematicians in England - India was a British colony at that time. The rest is history!

The two letters Ramanujan wrote to G. H. Hardy of Cambridge University are considered to be among the most significant in mathematical history. In these letters Ramanujan communicated hundreds of bewildering mathematical formulae he had discovered. Hardy and his peers in Cambridge, were convinced by the letters that Ramanujan was a genius of the class of Euler or Jacobi. Hardy invited Ramanujan to Cambridge to work with him so that the untutored genius could be given a proper sense

of direction. Orthodox Hindus believed that it was a sin to cross the oceans, and so Ramanujan declined this invitation; his mother would not give him permission to go. But Hardy persisted. One night, his mother had a dream in which she saw Ramanujan being honored by foreigners in a great assembly. In that same dream the Goddess of Namakkal ordered the mother not to stand in the way of her son's recognition! Thus with his mother's permission, Ramanujan sailed for England in 1914.

Hardy, being an agnostic, dismissed the Goddess of Namakkal stories as mere fables. However, I should point out that it is very natural for Hindus to accept such legends. Hindu belief isthat there is a divine origin to every aspect of knowledge, including music, and that is why much of Hindu classical music is devotional. Hindus believe in the story that Kalidasa, the greatest sanskrit poet, was transformed from an uneducated cowherd to a poet par-excellence overnight because the Goddess Kali wrote her blessings on his tongue. Thus it is very natural for Hindus to accept the Goddess of Namakkal as a divine origin of Ramanujan's great discoveries. In England, within a short span of five years, Ramanujan wrote several fundamental papers, some with Hardy, that revolutionized various areas of mathematics. But conditions were difficult in England at that time, worsened by the First World War. Ramanujan was a strict vegetarian, and food to suit his dietary needs was difficult to get in wartime England. He also did not take care to protect himself from the cold English winters. Thus he had to return to India in 1919, a very sick man. Hardy was concerned that Ramanujan might not live long, and so he worked hard to get him elected as Fellow of the Royal Society (FRS) in 1918. What a recognition for someone who did not even have a college degree! Shortly after Ramanujan returned to India, he died in April 1920 in Madras. In his 32 years, Ramanujan had made outstanding contributions, and was recognised with the highest honor that any academician in the British Commonwealth could aspire for, the election as Fellow of the Royal Society.

SASTRA University and Ramanujan's Home: The Shanmugha Arts, Science, Technology and Research Academy (SASTRA), is a private (deemed) university whose main campus is located in the town of Tanjore,

after which the district is named. SASTRA University was created about 15 years ago as a private educational institution. Unlike public colleges and universities in India, where most admission is based on a quota system for certain underdeveloped segments of society, admission to private educational institutions like SASTRA is based on merit. Thus in a short span of time, SASTRA attracted some of the brightest students and the best teachers, and therefore grew both in size and quality to attain the status of a deemed university. Owing to this successful growth, SASTRA recently opened a second campus at Ramanujan's hometown, Kumbakonam.

In 2003 SASTRA University purchased the home of Ramanujan, and will maintain it as a museum. Ramanujan is an idol and inspiration to all students in India, and hence the preservation of his home was essential to the spirit and hope of Indian intellectuals. SASTRA also created a Srinivasa Ramanujan Centre which has a library that contains several Ramanujan memorablia, as well as books, papers, and journals relating to his work. Since a university has now purchased Ramanujan's home, we now have the active involvement of administrators, academicians, and students, in the preservation of Ramanujan's legacy for posterity. To mark the occasion of the purchase of Ramanujan's home, the Srinivasa Ramanujan Centre of SASTRA University conducted an International Conference on Number Theory and Secure Communications during December 19-22, 2003, where I was invited to lecture on my work. Other plenary speakers at this conference included George Andrews from The Pennsylvania State University, Noam Elkies of Harvard University, Samuel Wagstaff of Purdue University, and Antol Balog of The Hungarian Academy of Sciences. George Andrews gave the Opening Lecture of the conference as well as the concluding Ramanujan Memorial Lecture on December 22, Ramanujan's birthday. The conference was supported by the Indo-US Forum, the Number Theory Foundation, and several funding agencies in India. The President of India, Dr. Abdul Kalam, inaugurated the conference and declared open Ramanujan's home as a museum and national treasure. My wife Mathura, who is a Bharathanatyam dancer and teacher, and I, were happy to have an opportunity to visit this home and the wonderful temples of the Tanjore district.

Accommodation and sightseeing: All the delegates with their families who came from overseas were accommodated at Sterling Resorts in the *Swami Malai* near Kumbakonam. Swami Malai derives its name from a temple there for Lord Subramanya, who as a child explained the deep meaning of the sacred Hindu syllable *Om* to his father Lord Shiva. The significance of this story is that knowledge has no age barriers. *Swami*, which means God, refers here to Lord Subramanya, and *Malai* which means mountain in Tamil, refers to a little hill on top of which the temple is located.

Sterling Resorts is an amazing place. It is a set of cottages with tiled roofs, in a farm or plantation setting. There are plenty of banana trees all around and several cows on the premises of the resort. Thus milk and all milk products required for guests are obtained fresh from the resort itself. Also, delicious Indian food is served Indian style on banana leaves. On the resort grounds there is a magnificent white statue of Lord Shiva with his young son Lord Subramanya on his lap whispering the meaning of *Om* into his father's ear. When all guests arrived at the resort, we were given a very warm traditional welcome, namely, we were not only greeted with garlands and flowers, but trained staff at the resort in traditional Indian dress, washed and massaged our feet with fragrant water. I suppose the tradition came about because in the past travellers used to arrive by foot, and so this foot massage was a welcome relief. We arrived by a van from Madras, yet we immensely enjoyed and appreciated this traditional welcome!

Although the Sterling Resorts has an old fashioned setting, the rooms have all amenities, including a TV, telephone, hot water heater, etc. There are no note pads near the telephone to take down messages. Instead, you are provided a slate and a piece of chalk. When I went to bed I hoped that like Ramanujan, I too would get a formula in my dream so that in the middle of the night I could get up and write it down on the slate with the piece of chalk!

Seeing Ramanujan's home was itself a dream come true. What an inspiration to see this small humble home from where so many significant

mathematical discoveries poured forth. The home has only three rooms surrounding a small courtyard - a bedroom, a kitchen, and a dining area. In the back there is a well and a bathroom. This was a typical village home where as was customary, a family of four to eight lived here. Not every one could sleep in the bedroom, and so members of Ramanujan's family slept in the courtyard as well. The home is located on Sarangapani Sannidhi Street, so named, because Sarangapani Temple is located at the end of the street. There is a window in the bedroom overlooking the street. As a boy Ramanujan used to sit on the bedroom cot doing mathematics and watching the passers by on the street through this window. Mathura and I had the pleasure of seeing Ramanujan's home along with Professor George Andrews of The Pennsylvania State University, one of the world's greatest authorities on Ramanujan's work.

After seeing Ramanujan's home, we went to the Sarangapani Temple to offer *Archana* to the Lord. Archana is a special form of prayer in which the priest mentions the family background (*sankalpa*) of the devotee on whose behalf he is offering the prayer. It was a very pleasant surprise that the high priest not only asked Professor Andrews to join us in the sanctum sanctorum of the temple but included his name in the sankalpa as well. In many Hindu temples of South India, even today, only Hindus are allowed into the inner sanctum. This recent practice of allowing others into the sanctum was a very pleasant development we saw in the Tanjore district at all temples we visited.

Tanjore town, which is about half an hour from Kumbakonam by car, has one of the most impressive temples in all of South India, the *Brihadeeswara temple* for Lord Shiva. When you enter its grounds you see a massive *Nandi*, or bull, which is the guardian deity in all Shiva temples. As we entered the Brihadeeswara temple, we saw an *abhisheka* (daily purification bath) being performed for the Nandi. When an abhisheka is performed, the diety is washed with milk, yoghurt, honey, and in this case also *veebothi* (holy ash of Lord Shiva), and finally with water. All through the abhisheka, the appropriate verses from the *vedas* (Hindu holy scriptures) are chanted. Imagine the amount of milk, yoghurt and honey used in the abhisheka of a massive Nandi!

After offering prayers at the Brihadeeswara temple, our hosts at SASTRA took us to a few shops in Tanjore town containing a wonderful collection of local art. Mathura wanted to purchase a Tanjore painting during a visit there. So we actually went to the home of a local artisan and bought a lovely painting of Lord Krishna. This now adorns the wall of the *puja* (worship) room in our home in Gainesville, to remind us every day of the rich cultural experience we had in Tanjore district. The visit was also truly memorable because it provided us an opportunity to see the home of Ramanujan, thereby making it an unforgettable mathematical pilgrimage.

About the Author:

Krishnaswami Alladi is Professor and Chairman of the Department of Mathematics at the University of Florida. His research is in Number Theory, an area where Ramanujan has made spectacular contributions. He is also the Editor-in-Chief of The Ramanujan Journal, an international publication devoted to all areas of mathematics influenced by Ramanujan. A shorter version of this article will appear in the newsletter of LOTUS (The Lord of the Universe Society), in Honolulu, Hawaii. Krishna's wife Mathura has given a Bharathanatyam (Indian classical dance) program at the East West Center, coorganized by the LOTUS Society in August 1993. They return to India every year but this was the first time they visited Kumbakonam to see Ramanujan's home and enjoy the rich cultural heritage of the Tanjore district.

Ramanujan's Notebooks

Published by:
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New York : Berlin : Heidelberg : London : Paris ; Tokyo ; Hong Kong : Barcelona : Budapest

Bruce C Berndt

Professor Emeritus
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green Street
Urbana, Illinois 61801-2975, USA
Mailing address:
273 Altgeld Hall
1409 W. Green St.
Urbana, IL 61801
berndt@math.illinois.edu

Research Interests

Classical analysis, in particular, as related to Ramanujan's notebooks, infinite series, elliptic and modular functions, theta-functions, q-series, continued fractions, character sums, special functions, asymptotic series, and contour integration.

Research Description

Analytic Number Theory
Ramanujan's Earlier Notebooks and Lost Notebook
Special Functions
Elliptic (especially Theta) Functions
q-series
Continued Fractions

Part – I

[368 Pages]

To his wife Helen and their children Kristin, Sonja, and Brooks

Please visit to read:

https://archive.org/details/ramanujans-notebooks/Ramanujan%20Notebooks%20I/

Introduction

Srinivasa Ramanujan occupies a central but singular position in mathematical history. The pathway to enduring, meaningful, creative mathematical research is by no means the same for any two individuals, but for Ramanujan, his path was strewn with the impediments of poverty, a lack of a university education, the absence of books and journals, working in isolation in his most creative years, and an early death at the age of 32. Few, if any, of his mathematical peers had to encounter so many formidable obstacles.

Contents

Introduction

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Magic Squares

CHAPTER 2
Sums Related to the Harmonic Series or the Inverse Tangent Function

CHAPTER 3
Combinatorial Analysis and Series Inversions

CHAPTER 4
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CHAPTER 5
Eulerian Polynomials and Numbers, Bernoulli Numbers, and the Riemann Zeta-Function

CHAPTER 6
Ramanujan's Theory of Divergent Series

CHAPTER 7
Sums of Powers, Bernoulli Numbers, and the Gamma Function

CHAPTER 8
Analogues of the Gamma Function

CHAPTER 9
Infinite Series Identities, Transformations, and Evaluations

Ramanujan's Quarterly Reports References

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Part - II

[372 Pages]

Dedicated to his mother Helen and the memory of his father Harvey

Please visit to read:

https://archive.org/details/ramanujans-notebooks/Ramanujan%27s%20Notebooks%20II/

Introduction

We take up something-——we know it is finite; but as soon as we begin to analyze it, it leads us beyond our reason, and we never find an end to all its qualities, its possibilities, its powers, its relations. It has become infinite.

‹‹‹‹‹Vivekananda

In a certain sense, mathematics has been advanced most by those who are distinguished more for intuition than for rigorous methods of proof.

‹‹‹‹‹‹ Felix Klein

For now, we see through a glass, darkly; but then face to face: now I know in part; but then shall I know even as also I am known.

‹‹‹‹‹‹‹ First Corinthians 13:12

The quoted passages of Vivekananda, Klein, and St. Paul each point to a certain facet of Ramanujan's work. First, on June 1—5, 1987, the centenary of Ramanujan's birth was celebrated at the University of Illinois with a series of 28 expository lectures and several contributed papers that traced Ramanujan's influence to many areas of current research. Thus, Ramanujan's mathematics

continues to generate a vast amount of research in a variety of areas. Second, in the sequel, we shall see many instances where Ramanujan made profound contributions but for which he probably did not have rigorous proofs; for example, see Entry 10 of Chapter 13. Third, although St. Paul's passage is eschatological in nature, it points to the great need to learn how Ramanujan reasoned and made his discoveries. Perhaps we can prove Ramanujan's claims, but we may not know the well from which they sprung. These three aspects of Ramanujan's work will frequently be made manifest in the pages that follow.

Contents

Preface Introduction

CHAPTER 10
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CHAPTER 11
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References

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Part – III

[524 Pages]

Dedicated to S. Janaki Ammal (Mrs. Ramanujan)

Please visit to read:

https://archive.org/details/ramanujans-notebooks/Ramanujan%27s%20Notebooks%20III/

Introduction

In der Theorie der Thetafunctionen ist es leicht, eine beliebig grosse Menge von Relationen aufzustellen, aber die Schwierigkeit beginnt da, wo es sich darum handelt, aus diesem Labyrinth von Formeln einen Ausweg zu finden.

<<<<< G. Frobenius

The content of this volume is more unified than those of the first two volumes of our attempts to provide proofs of the many beautiful theorems bequeathed to us by Ramanujan in his notebooks. Theta-functions provide the binding glue that blends Chapters 16—21 together. Although we provide proofs here for all of Ramanujan's formulas, in many cases, we have been unable to find the roads that led Ramanujan to his discoveries. It is hoped that others will attempt to discover the pathways that Ramanujan took on his journey through his luxuriant labyrinthine forest of enchanting and alluring formulas.

Contents

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CHAPTER 16
q-Series and Theta-Functions

CHAPTER 17
Fundamental Properties of Elliptic Functions

CHAPTER 18
The Jacobian Elliptic Functions

CHAPTER 19
Modular Equations of Degrees 3, 5, and 7 and Associated Theta-Function Identities

CHAPTER 20
Modular Equations of Higher and Composite Degrees

CHAPTER 21
Eisenstein Series

References

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Part – IV

[231 Pages]

Dedicated to the People of India

Please visit to read:
https://archive.org/details/ramanujans-notebooks/Ramanujan%27s%20Notebooks%20IV/

Introduction

If you have built castles in the air, your work need not be lost; that is where they should be. Now put the foundations under them.

‹‹‹‹‹ H D Thoreau - *Walden*

Ramanujan built many castles. Although some may have been lost,
most were preserved. Since his death in 1920,
many mathematicians have been constructing
the foundations for these magnificent structures.
We continue this task in the present volume.

Preface
Introduction

CHAPTER 22
Elementary Results

CHAPTER 23
Number Theory

CHAPTER 24
Ramanujan's Theory of Prime Numbers

CHAPTER 25
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CHAPTER 26
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CHAPTER 30
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CHAPTER 31
Elementary and Miscellaneous Analysis

Location in Notebook 2 of the Material in the 16 Chapters of Notebook 1

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Part – V

[433 Pages]

[George E. Andrews · Bruce C. Berndt]

Please visit to read:
https://www.pdfdrive.com/ramanujans-lost-notebook-part-v-e187537428.html

Preface

This is the fifth and final volume that the authors have written in their examination of all the claims made by S. Ramanujan in The Lost Notebook and Other Unpublished Papers. Published by Narosa in 1988, the treatise contains the "Lost Notebook," which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included in this publication are partial manuscripts, fragments, and letters from Ramanujan to G.H. Hardy. In his last letter, Ramanujan introduced mock theta functions to the mathematical world for the first time. Most of this volume is devoted to Ramanujan's beautiful identities involving mock theta functions, which populate his "Lost Notebook." Also featured are Ramanujan's many elegant Euler

products, found in scattered entries and in a manuscript published with the "Lost Notebook." A few continued fractions are also examined.

1. Introduction

2 Third Order Mock Theta Functions: Elementary Identities

3. Fifth Order Mock Theta Functions: Elementary Identities

4. Third Order Mock Theta Functions: Partial Fraction Expansions

5. The Mock Theta Conjectures: Equivalence

6. Fifth Order Mock Theta Functions: Proof of the Mock Theta Conjectures

7. Sixth Order Mock Theta Functions

8. Tenth Order Mock Theta Functions: Part I, The First Four Identities

9. Tenth Order Mock Theta Functions: Part II, Identities for $\phi 10(q)$, $\psi 10(q)$

10. Tenth Order Mock Theta Functions: Part III, Identities for $\chi 10(q)$, $X10(q)$

11. Tenth Order Mock Theta Functions: Part IV

12. Transformation Formulas: 10th Order Mock Theta Functions

13. Two Identities Involving a Mordell Integral and Appell–Lerch Sums

14. Ramanujan's Last Letter to Hardy

15. Euler Products in Ramanujan's Lost Notebook

16. Continued Fractions

17. Recent Work on Mock Theta Functions

18. Commentary on and Corrections to the First Four Volumes

19. The Continuing Mystery

His Recent Publications

01] Berndt, B. C., Fassina, M., Kim, S., & Zaharescu, A. (2022). Two-parameter identities for divisor sums in algebraic number fields. *Journal of Mathematical Analysis and Applications*, *506*(2), [125679]. https://doi.org/10.1016/j.jmaa.2021.125679

02] Alladi, K., Berndt, B. C., Paule, P., Sellers, J. A., & Yee, A. J. (2021). Preface. *George E. Andrews: 80 Years of Combinatory Analysis*, v-vi. (Trends in Mathematics). Springer. https://doi.org/10.1016/B978-0-08-031874-5.50006-1

03] Berndt, B. C. (2021). Ramanujan, His Lost Notebook, Its Importance. In *George E. Andrews: 80 Years of Combinatory Analysis*, pp. 33-52. (Trends in Mathematics). Springer. https://doi.org/10.1007/978-3-030-57050-7_3

04] Berndt, B. C., Li, J., & Zaharescu, A. (2021). The Final Problem: A Series Identity from the Lost Notebook. In *George E. Andrews: 80 Years of Combinatory Analysis*, pp. 783-790. (Trends in Mathematics). Springer. https://doi.org/10.1007/978-3-030-57050-7_42

05] Andrews, G. E., Berndt, B. C., Chan, S. H., Kim, S., & Malik, A. (2020). Four Identities for Third Order Mock Theta Functions. *Nagoya Mathematical Journal*, *239*, 173-204. https://doi.org/10.1017/nmj.2018.35

Ramanujan Museum

https://casualwalker.com/museum-for-the-man-who-knew-infinity-ramanujan-museum-royapuram-chennai-travel-guide/

Srinivasa Ramanujan

The Man Who Knew Infinity, was one of the World's greatest mathematical geniuses. Ramanujan had no formal training in pure mathematics, but he made great contributions to the analytical theory of numbers, infinite series and continued fractions, including solutions to mathematical problems which are considered unsolvable.

To honour this world renounced mathematical genius, Ramanujan Museum was established in 1993 in Chennai by Mr P.K. Srinivasan a Math educator who spent

almost 25 years collecting the resources which celebrate Srinivasa Ramanujan's life and massive contributions to the world of mathematicians. The museum was accommodated in the premises of the Avvai Cultural Academy, Royapuram, Chennai by Mr A. T. Bose who is currently head and leading the Ramanujan Museum along with its director Ms Meena Suresh.

To celebrate Ramanujan's birthday on December 22nd, each year the Ramanujan Museum organizes an annual lecture by eminent mathematicians. The Centre also has programmes to spread awareness of mathematics among children.

This museum treasures the pictures, letters and documents focusing on the greatest mathematician of the 20th Century. We can find the exhibits of numerous Ramanujan memorabilia, including photographs of the Ramanujan's home and family, Ramanujan's horoscope, Ramanujan's Mother Komalattammal, Ramanujan's wife Janaki, Ramanujan's early school/ Pachaiyappa's College days, Awards he received during his school time, his days at Trinity College – Cambridge, Ramajuman's three-part / volume notebooks, Ramanujan's various mathematical models, formulas and theorems, Indian Mathematical Society supporters, Ramanujan's postal stamp to commemorating his 75th Birth Anniversary, Ramanujan's correspondence with friends, relatives, and colleagues, as well as his passport, Ramanujan's handwritten job

application for the post of a clerk at Madras Port Trust, Hardy's replies to Ramanujan's letters sent to him in early 1913. His supporters from early to last days, his handwritten letters and mathematical notebooks and much more Ramajuman's exiting work to be seen live and direct at the Museum.

****

Address: Ramanujan Museum & Math Education Centre, 15/9, Somu Chetty, 4th Lane, Royapuram, Chennai, Tamil Nadu 600013. India

Hours: 10 AM to 7.30 PM (All working days)
Phone: 044 2596 0877 / **Email:** Ramanujanmuseum@yahoo.com

Reaching:

On Road: Taxis, buses and hired cars are easy ways to get around

Nearest Railway Station: Chennai Central and Chennai Egmore Railway station

Nearest Airport: Chennai International Airport

Please visit the web link to enjoy photo captures of Ramanujan Museum, Royapuram in Chennai.

https://casualwalker.com/museum-for-the-man-who-knew-infinity-ramanujan-museum-royapuram-chennai-travel-guide/

The Ramanujan Math Park

[https://gyanome.org/ramanujan-maths-park/]

[https://www.agastya.org/]

The Ramanujan Math Park is located inside Agastya Campus Creativity Lab, Kuppam AP, at the intersection of the states of Karnataka, Andhra Pradesh, and Tamil Nadu, in Southern India. It is a unique outdoor and indoor mathematical experience with large scale exhibits and interactive touch screen stations. The Math Park is a project by Gyanome non-profit organisation (www. gyanome.org) in collaboration with Agastya

Agastya International Foundation, in association with Gyanome Foundation, Imaginary and SBI Funds Management Private Ltd. the investment manager of SBI Mutual Fund, is inaugurating the Ramanujan Math Park on December 22, 2017, at its 172-acre Creativity Campus situated in Kuppam.

Agastya Campus covers an area of 172 acres and is visited by more than 500 students per day, whose curiosity is sparked by Agastya's hands-on teaching methods. Children are bussed in for lessons in science, math, and art, which augment the curriculum in the government schools. Agastya's instructors have been able to improve upon their methods of educating children and to develop programs for training local teachers, as well. Camps are arranged for groups from schools in faraway cities, providing urban children with a rare opportunity to experience nature close up, in a rural environment. The campus also attracts teachers from private schools all over India, who come here to learn ways to infuse their own teaching with the Agastya spark.

Having drawn inspiration from the extraordinary works of Indian Mathematician, Srinivasa Ramanujan, the park aims to provide a space to promote learning, exploration, and discovery of ideas that convey a sense of the ubiquity of Math. By giving hands-on learning experience of math concepts, it aims to increase enthusiasm and reduce and eliminate obstacles in learning math, especially among economically underprivileged children.

VISION

The Ramanujan Math Park is envisaged as an evolving platform for students, teachers and math enthusiasts to come together and communicate various mathematical ideas that will lead to greater engagement with the subject.

MISSION

- To ignite curiosity and interest in mathematics by focusing on creating content that scaffolds as learning materials in schools, while being in lieu to the curriculum.
- To facilitate the learning of math concepts by developing exhibits, pedagogies and engaging activities.
- To instill a sense of appreciation and excitement in children by tapping into the mysteries, beauty and relevance of math.
- To make the learning of math contextual, conceptual and meaningful.

OBJECTIVES

- ✓ To introduce the all-pervasive nature of mathematics by focusing on evolution, storytelling, and utility of various mathematical concepts.
- ✓ To facilitate a space for understanding the fundamentals of math, using interactive hands-on approach through the display of exhibits that demonstrate connections between math and various other fields like architecture, music, arts, astronomy, physics, engineering, technology etc.
- ✓ To encourage and engage students and teachers to create content that is relevant to the Indian context and demonstrate their understanding of math by way of essays, competitions, etc.
- ✓ To inspire students by introducing and featuring the works of Indian mathematicians.
- ✓ To highlight the contributions of India and other civilizations in the evolution of mathematical ideas.
- ✓ To create digital content that aims at empowering teachers with teaching resources that helps them to engage and use technology meaningfully in classrooms.
- ✓ To document and preserve the existing mathematical knowledge present in rural India.
- ✓ To create a space where experts and reputed institutions from all over the world participate and support in evolving ideas, through talks, training, performances, etc.

The Ramanujan Math Park will focus on creating a learning environment that fosters and nurtures the desire to engage with mathematics in enriching ways, amongst broad strata of the society.

For VIRTUAL TOUR, please visit the web link:

https://imath.iiitb.ac.in/ramanujan-math-park/

Srinivasa Ramanujan Centre

Kumbakonam

SASTRA's off campus centre at Kumbakonam known as Srinivasa Ramanujan Centre (SRC) is a fitting tribute to the great Mathematician, Srinivasa Ramanujan, who spent most of his formative years in this temple town of Kumbakonam. Kumbakonam gets its name from the presiding deity Lord Aadhi Kumbeshwarar. The cradle of South Indian Culture, the town is known for its temples, intricately carved panchaloha idols, exquisite brasswares especially lamps, silk and betel leaves. The place, like many other sacred places spread across the country, is known for the Mahamaham festival. The festival celebrated once in twelve years coincides with the entry of Sun and Jupiter into constellation of Aquarius and Leo respectively. People from all walks of life have a dip in the Mahamaham tank along with the presiding deities of the town. This occasion is said to absolve the people of their sins.

Many intellectuals, musicians, sthapathis (people well-versed in design and construction of temples and carving of idols) and silk weavers hail from in and around Kumbakonam. Located on the banks of river Cauvery, this place can be reached from Chennai, either by rail or by road in about six hours.

SASTRA

The dreams of its founding fathers took shape in 1984 in the form of Shanmugha College of Engineering, which is now renamed, SHANMUGHA ARTS, SCIENCE, TECHNOLOGY & RESEARCH ACADEMY (SASTRA). Our programmes, infrastructure, teaching-learning, etc., are among the country's finest. SASTRA has always been proactive in its academic planning and believes in gaining the first-mover advantage. As a testimony to this, the Government of India has conferred the University status to SASTRA on April 26, 2001 under section 3 of the UGC Act, 1956.

SASTRA offers various undergraduate and post graduate courses in Engineering, Science, Education, Management, Law and Arts besides various Doctoral programmes and has state-of-the-art laboratories, a well-stocked library and one of the best computing facilities. With a sprawling campus having a built-up area of 2,22,129.02

square metre spread over 232 acres and a vibrant population of over 10,000 students and over 700 teaching faculty have made SASTRA a landmark in the educational map of India.

With an ideal teacher-taught ratio, we strive for academic excellence through personalized attention. The mechanisms established to support and monitor the student's progress assure success and satisfaction. Since its inception SASTRA has achieved national standing in terms of academic performance, co-curricular and extra-curricular activities and also in its growth and commitment to social service.

The standards of excellence of our programmes are reflected in the grades awarded to us by the National Assessment and Accreditation Council and in the success of our students in industry and academia, both in India & abroad. SASTRA has also been re-accredited with Grade 'A' (maximum) by the National Accreditation and Assessment Council (NAAC) a statutory body of U.G.C.

SASTRA shapes its students' future by fostering a teamwork approach to instruction, encouraging interaction with faculty, providing access to high tech information, motivating them to develop new ideas and concepts, taking personal interest in students' career development and preparing them for success.

SRC

Srinivasa Ramanujan Centre (SRC), Kumbakonam, was inaugurated, in November 2000, by Prof. .S. Ramamurthy, Secretary, Department of Science and Technology, Government of India with the following objectives:

To establish a national monument for Srinivasa Ramanujan

- To provide quality higher education to the people of Kumbakonam and nearby places
- To offer degree programmes
- To coach +2 students for professional courses

The activities of the Srinivasa Ramanujan Centre have been expanded with SASTRA acquiring 10 acres of land in Lakshmivilas Agraharam and constructing a massive six storied structure encompassing an area of 1,52,523 sq.ft. Situated in the heart of the

town, this building houses modern class rooms, state-of-the-art Computer laboratory, Electrical Machines laboratory, Electronics and Communications laboratory, Physics laboratory, Chemistry and Biochemistry laboratory, Mechanical Workshops and a 300 - seater airconditioned auditorium. A library with over 21,000 books and more than 145 print journals in addition to a free 2Mbps Internet access caters to the requirements of students and staff members. House of Ramanujan Mathematics, a museum on life and works of the Mathematical prodigy, Srinivasa Ramanujan, also exists on this campus. SRC has been dedicated to the Nation by His Excellency Dr. A.P.J. Abdul Kalam, President of India.

Based on the recommendations of a high-level committee appointed by the University Grants Commission (UGC), Government of India, **Srinivasa Ramanujan Centre,** established by SASTRA has been declared as an OFF-CAMPUS CENTRE under the ambit of SASTRA University.

Major Activities

- ✓ SASTRA purchased the house where Srinivasa Ramanujan lived at Kumabakonam and renovated it. The President of India, His Excellency Dr. A.P.J. Abdul Kalam has since declared it as an international Monument.
- ✓ The President dedicated the House of Ramanujan Mathematics, a museum containing life and works of the prodigy to the Nation.
- ✓ SASTRA organizes an International Conference, every year, in honour of Srinivasa Ramanujan on various focal themes.
- ✓ SASTRA organizes Srinivasa Ramanujan Birth Anniversary Commemorative lecture every year on Dec 22, 2012.
- ✓ SASTRA has instituted an annual award of US$ 10,000/- and a citation to be presented for outstanding research in the areas influenced by Srinivasa Ramanujan. The award is open to researchers all over the world under the age of 32. This award has been instituted with a view to encourage young minds to pursue cutting edge research in mathematical applications

influenced by Ramanujan and increase the visibility of mathematics both at the national and global level.

SASTRA-Ramanujan Conference

- The Seventeenth International conference on Number Theory was held on December 21 & 22, 2019. Prof. Ken Ono, University of Virginia, USA, inaugurated the conference.
- The Sixteenth International conference on Srinivasa Ramanujan was held on December 21 & 22, 2018. Dr. C. Velan, Executive Director & CEO, Ramanujan IT City, Tata Realty Infopark, Ltd., inaugurated the conference.
- The Fifteenth international conference on Number Theory was held on December 21 & 22, 2017. Mr. Badrinarayanan Parthasarathy Vice-president, Networks & Services Engineering, Tata Communications, inaugurated the conference.
- The Fourteenth international conference on Number Theory was held on December 21 & 22, 2016. Prof. Henri Darmon, Department of Mathematics, McGill University, Canada, inaugurated the conference.
- The Thirteenth international conference on Number Theory and Algebraic Geometry was held on December 21 & 22, 2015. Prof. G. Rangarajan, Department of Mathematics and Chairman, Division of Interdisciplinary Research, IISc., Banglore, inaugurated the conference.
- The Twelfth international conference on Number Theory was held on December 21 & 22, 2014. Dr. Mangalam Srinivasan, Former Advisory, Research and Teaching appointee at the faculty of Arts and Sciences, Special Advisor, Kennedy School of Government and Fellow of CFTA, Harvard University inaugurated the conference.
- The Eleventh international conference on Number Theory and Galois Representations was held on December 21 & 22, 2013. Prof. Michael Rapoport, Mathematisches Institut, Uuniversity of Bonn, Germanay, inaugurated the conference.
- The Tenth international conference on Legacy of Srinivasa Ramanujan was held on December 14 & 15, 2012 at Srinivasa Ramanujan Centre, Kumbakonam. Dr. Rajiv Sharma, Executive Director, Indo-US Science and Technology Forum, New Delhi, inaugurated the conference.
- The Nineth conference on "Number Theory and Automorphic Forms" was held on December 22, 2011. Conference Topic : Ergodic Theory and Dynamics.
- The Eighth conference on "Number Theory and Automorphic Forms" was held on December 22, 2010 inaugurated by Prof. Lakshmi Narayanan,

Senior Professor, Raman Research Institute, Bangalore. Topic : Number Theory and Automorpic Forms.

- The Seventh conference on "Number Theory and Mock Theta Function" was held on December 22, 2009 inaugurated by Prof. M. Vijayan, Honorary Professor, IISC Bangalore. Topic : Number Theory and Mock Theta Function.
- The Sixth conference on "Number Theory and Modular Forms" was held from December 20 to 22, 2008 inaugurated by Dr. Krishna N Ganesh, Director IISER Pune & J C Bose Fellow(DST), NCL Pune. Conference Topic : Number Theory and Modular Forms.
- The Fifth conference on "Number Theory, Theoretical Physics and Special functions" was held from December 20 to 22, 2007 inaugurated by Prof. Michel Waldschmidt. Conference Topic : Theoritical Physics and Special Functions.
- The Fourth conference was held from December 19 to 22, 2006 on "Number Theory and Combinatorics" was inaugurated by Shri. S. Sathyam, IAS(Retd), MHRD, Delhi. Conference Topic : Number Theory & Combinatorics.
- The Third in the series on "Number Theory and Mathematical Physics" was inaugurated by Dr. Arabinda Mitra, Executive Director, INDO-US Science and Technology Forum, New Delhi, on December 20, 2005. Conference Topic : International Conference on Number Theory & Mathematical Physics.
- The Second conference on "Number Theory and Fourier Techniques" was inaugurated by Dr. Alladi Krishnaswamy, Professor and Chair, Department of Mathematics, University of Florida, on 20th December 2004. Conference Topic : Number Theory and Fourier Techniques.
- The First conference on "Number Theory for Secure Communications" was organized on 20th December 2003. It was inaugurated by His Excellency, Dr. A.P.J. Abdul Kalam, President of India. Conference Topic : International Conference on Number Theory for Secure Communications.

SASTRA Ramanujan Award

SASTRA has instituted an annual award of US$ 10,000/- and a citation to be presented for outstanding research in the areas influenced by Srinivasa Ramanujan. The award is open to researchers all over the world under the age of 32. This award has been instituted with a view to encourage young minds to pursue cutting edge research in mathematical applications influenced by Ramanujan and increase the visibility of mathematics both at the national and global level.

Yunqing Tang receives the Award for 2022

- The SASTRA Ramanujan Prize for **2022** was awarded to Yunqing Tang Assistant Professor at the University of California, Berkeley, USA.
- The **2021** SASTRA Ramanujan Prize was awarded to Dr. Will Sawin of Columbia University for his many path-breaking contributions at the interface of number theory and algebraic geometry utilizing a variety of powerful methods from different areas of mathematics.
- Sastra University (Deemed) has awarded the **2020** Sastra Ramanujan prize to Dr. SHAI EVRA of Princeton University, USA, and the Hebrew University of Jerusalem, Israel.
- The SASTRA Ramanujan prize for **2019** was awarded to mathematician Dr.Adam Harper, Assistant Professor with the University of Warwick, England.
- The **2018** SASTRA Ramanujan Prize was jointly awarded to Professors YIFENG LIU (Yale University, USA) and Jack Thorne (Cambridge University, England).
- Maryna Viazovska of Swiss Federal Institute of Technology, Lausanne, Switzerland, has received **2017** SASTRA Ramanujan Prize for her contribution to number theory.
- The **2016** SASTRA Ramanujan prize, for outstanding contributions by young mathematicians to areas influenced by the genius Srinivasa Ramanujan, was jointly awarded to Kaisa Matomaki of University of Turku, Finland and Maksym Radziwill of McGill University, Canada.
- The **2015** SASTRA Ramanujan Prize was awarded to Dr. Jacob Tsimerman of the University of Toronto, Canada.
- The **2014** SASTRA Ramanujan Prize was awarded to Dr. James Maynard of Oxford University, England, and the University of Montreal, Canada.
- The **2013** SASTRA Ramanujan Prize was awarded to Professor Peter Scholze of the University of Bonn, Germany.

- The **2012** SASTRA Ramanujan Prize was awarded to Professor Zhiwei Yun, who has just completed a C. L. E. Moore Instructorship at the Massachusetts Institute of Technology and is taking up a faculty position at Stanford University in California this fall.
- The **2011** SASTRA Ramanujan Prize was awarded to Roman Holowinsky, who is now an Assistant Professor at the Department of Mathematics, Ohio State University, Columbus, Ohio, USA.
- The **2010** SASTRA Ramanujan Prize was awarded to Wei Zhang, who is now a Benjamin Pierce Instructor at the Department of Mathematics, Harvard University, USA.
- The **2009** SASTRA Ramanujan Prize will be awarded to Professor KATHRIN BRINGMANN of the University of Cologne, Germany and the University of Minnesota, USA.
- The **2008** SASTRA Ramanujan Prize will be awarded to Akshay Venkatesh, who is now Professor of Mathematics at Stanford University, USA.
- The **2007** SASTRA Ramanujan Prize will be awarded to Ben Green, who is Hershel Smith Professor of Mathematics at Cambridge University, England.
- The **2006** SASTRA Ramanujan Prize will be awarded to Professor Terence Tao of the University of California at Los Angeles (UCLA).
- The **2005** SASTRA Ramanujan Prize will be jointly awarded to Professors MANJUL BHARGAVA (Princeton University) and KANNAN SOUNDARARAJAN (University of Michigan).

Prof Manjul Bhargava and Prof Kannan Soundararajan receive the Award for 2005

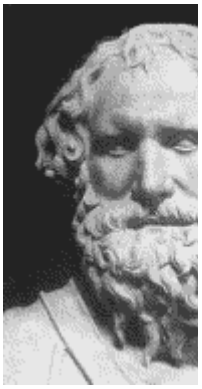
The

Hundred Greatest Mathematicians of the Past

This is the long page, with list and biographies.

(Click here for just the List, with links to the biographies. Or Click here for a List of the 200 Greatest of All Time.)

Isaac Newton


Archimedes

Carl Gauss

Leonhard Euler

Bernhard Riemann

David Hilbert

J.-L. Lagrange

Euclid

Alex. Grothendieck

G.W. Leibniz

John von Neumann

Henri Poincaré

Ranked in approximate order of "greatness."

Listed: The Mathematicians were born before 1930 and their work have **breadth, depth**, and **historical importance**.

https://fabpedigree.com/james/mathmen.htm

The 10 best mathematicians

Selected Maths Geniuses whose revolutionary discoveries changed our world.

Visit to know:

https://www.theguardian.com/culture/2010/apr/11/the-10-best-mathematicians

****

Greatest Mathematicians of All Time
List of Top 11 Mathematicians

Let's just admit that Mathematics is one of the most fun and popular subjects. People love Mathematics for various reasons. However, there are many who may feel slightly afraid thinking that it is way too confusing. Well, it is not. For people who have a hood teacher and guidance, Maths can become like a cakewalk. All you need to do is have some idea about the concepts of Mathematics. Following the life of the greatest mathematicians of all time can help you learn a lot.

Maths has had a contribution to the development of society as well. Most of the scientific discoveries were possible because of Maths. It allowed us to know the secrets of DNA, create and transmit electricity and also brought computers into existence. We owe the success and all these contributions to the greatest mathematician of all time. Read the article further to know some of the greatest Mathematicians.

The greatest mathematicians of all time have contributed significantly to the growth of the world. Most of them had also received awards and acknowledgements for their contribution to the field of Maths. Below is a list of all the greatest mathematicians of all time.

Please visit to study them:

https://www.embibe.com/exams/11-greatest-mathematicians-of-all-time/

1. Carl Friedrich Gauss

Born: April 30, 1777, Braunschweig, Germany
Died: February 23, 1855, Göttingen, Germany
Awards: Copley Medal
Education: University of Helmstedt, University of Göttingen, Braunschweig University of Technology

Carl Friedrich Gauss was a German mathematician who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, mechanics, electrostatics, astronomy, and matrix theory, and optics.

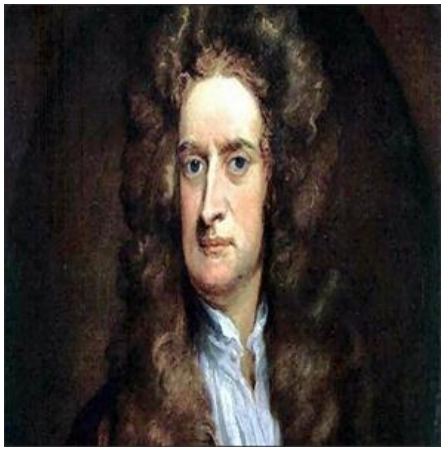
2. Leonhard Euler

Born: April 15, 1707, Basel, Switzerland
Died: September 18, 1783, Saint Petersburg, Russia
Education: University of Basel (1720–1723)
Influenced: Carl Friedrich Gauss, Joseph-Louis Lagrange, and more
Spouse: Salome Abigail Gsell (m. 1776–1783), Katharina Gsell (m. 1734–1773)

Leonhard Euler was a Swiss mathematician, physicist, astronomer, logician and engineer who made important and influential discoveries in many branches of mathematics like infinitesimal calculus and graph theory while also making pioneering contributions to several branches such as topology and analytic number theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function.

3. Isaac Newton

Born: January 4, 1643, Woolsthorpe-by-Colsterworth, United Kingdom
Died: March 31, 1727, Kensington, London, United Kingdom
Full name: Sir Isaac Newton
Education: Trinity College, Cambridge (1667–1668), The King's School, Grantham (1955-1959)



Sir Isaac Newton PRS was an English mathematician, astronomer, and physicist who is widely recognised as one of the most influential scientists of all time and a key figure in the scientific revolution.

4. Euclid

Born: Alexandria, Egypt
Nationality: Greek
Influenced: Stilpo, Thrasymachus of Corinth, Clinomachus, Eubulides, Ichthyas
Teacher: Archimedes
Field: Mathematics

Euclid of Megara was a Greek Socratic philosopher who founded the Megarian school of philosophy. He was a pupil of Socrates in the late 5th century BCE and was present at his death. He held the supreme good to be one, eternal and unchangeable and denied the existence of anything contrary to the good.

5. Srinivasa Ramanujan

Born: December 22, 1887, Erode
Died: April 26, 1920, Kumbakonam
Spouse: Janakiammal (m. 1909–1920)
Education: Trinity College, Cambridge (1919–1920), University of Cambridge (1914-1919)
Parents: Komalatammal, K. Srinivasa Iyengar

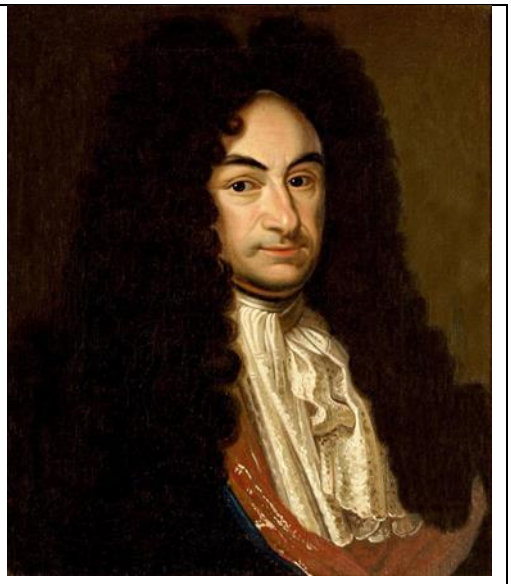
Srinivasa Ramanujan was an Indian mathematician who made significant contributions to mathematical analysis, number theory and continued fractions. At age 31 Ramanujan was one of the youngest Fellows in the history of the Royal Society. He was elected "for his investigation in Elliptic functions and the Theory of Numbers." On 13 October 1918, he was the first Indian to be elected a Fellow of Trinity College, Cambridge. He is considered to be one of the best Indian mathematicians.

6. Pierre de Fermat

Born: August 17, 1601, Beaumont-de-Lomagne, France
Died: January 12, 1665, Castres, France
Education: University of Orléans (1623–1626)
Spouse: Louise Long Fermat (m. ?–1665)
Books: Writings on Geometrical Loci
Parents: Dominique Fermat, Françoise Cazeneuve Fermat

Pierre de Fermat, A French mathematician who is often called the founder of the modern theory of numbers. Fermat developed a system of analytic geometry which both preceded and surpassed that of Descartes; he developed methods of differential and integral calculus which Newton acknowledged as an inspiration. He was also the first European to find the integration formula for the general polynomial, he used his calculus to find centres of gravity etc.

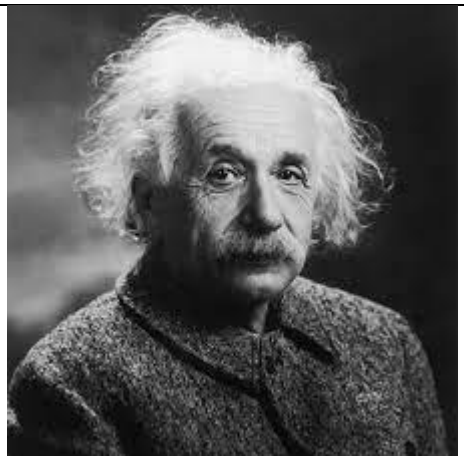
7. Gottfried Wilhelm Leibniz



Born: July 1, 1646, Leipzig, Germany
Died: November 14, 1716, Hanover, Germany
Influenced: Ferdinand Georg Frobenius, more
Education: Leipzig University, University of Altdorf, University of Jena
Influenced by: René Descartes, Baruch Spinoza, Blaise Pascal and many more.

Leibniz pioneered the common discourse of mathematics, including its continuous, discrete, and symbolic aspects. His ideas on symbolic logic weren't pursued and it was left to Boole to reinvent this almost two centuries later.

8. Albert Einstein

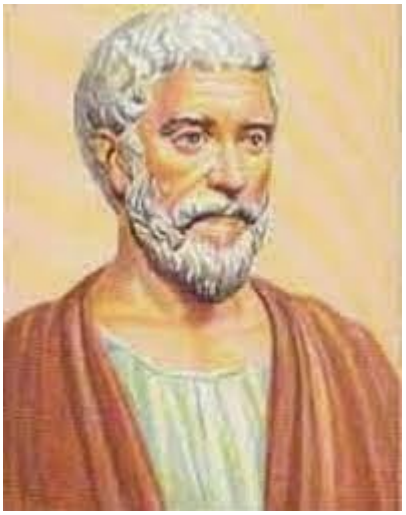


Born: March 14, 1879, Ulm, Germany
Died: April 18, 1955, Princeton, New Jersey, United States
Influenced: Satyendra Nath Bose, Leo Szilard, Wolfgang Pauli, and more
Influenced by: Isaac Newton, Mahatma Gandhi and many more.

Albert Einstein was unquestionably one of the two greatest physicists in all of history. The atomic theory achieved general acceptance only after Einstein's 1905 paper which showed that atoms' discreteness explained Brownian motion. Another 1905 paper introduced the famous equation $E = mc^2$. Einstein published at least 300 books or papers on physics altogether.

9. Pythagoras

Areas of interest: Politics, Mathematics, Metaphysics, Ethics, Music
Influenced: Philolaus, Empedocles, Plato, Alcmaeon of Croton, Euclid, Johannes Kepler, Parmenides, Hippasus
Philosophical era: Ancient philosophy
Schools of thought: Pythagoreanism
Influenced by: Thales of Miletus, Anaximander, Pherecydes of Syros, Themistocles



Pythagoras discovered that harmonious intervals in music are based on simple rational numbers. This led to a fascination with integers and mystic numerology. The Pythagorean Theorem was known long before Pythagoras, but he is often credited with the first proof. Apastambha proved it in India at about the same time; some conjecture that Pythagoras journeyed to India and learned of the proof there.

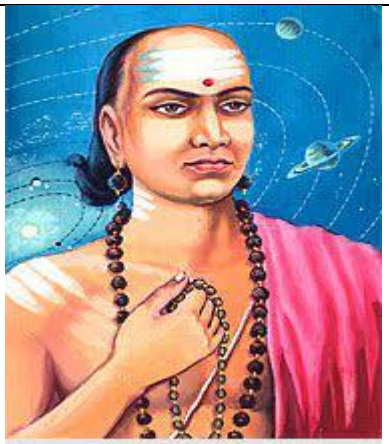
10. René Descartes

Born: March 31, 1596, Descartes, Indre-et-Loire, France
Died: February 11, 1650, Stockholm, Sweden
Influenced: Noam Chomsky, Baruch Spinoza, Slavoj Žižek, more
Influenced by: Aristotle, Plato, Thomas Aquinas, Archimedes and many more.

René Descartes is considered the inventor of both analytic geometry and symbolic algebraic notation. His use of equations to partially solve the geometric problem of Pappus revolutionized mathematics.

11. Aryabhata

Born: 476 AD, Assaka
Died: 550 AD, India
Nationality: Indian
Books: Aryabhatiya
Main interests: Mathematics, Astronomy
Influenced: Lalla, Bhāskara I, Brahmagupta, Varāhamihira


Aryabhata

Indian mathematicians excelled for thousands of years, and eventually even developed advanced techniques like the Taylor series before Europeans did, but they are denied credit because of Western ascendancy. Among the Hindu mathematicians, Aryabhata known as Arjehir by Arabs, maybe the most famous. Aryabhata is famous for the identity Σ (k3) = (Σ k)2. His most famous accomplishment in mathematics was the Aryabhata Algorithm for solving Diophantine equations. He made several important discoveries in astronomy.

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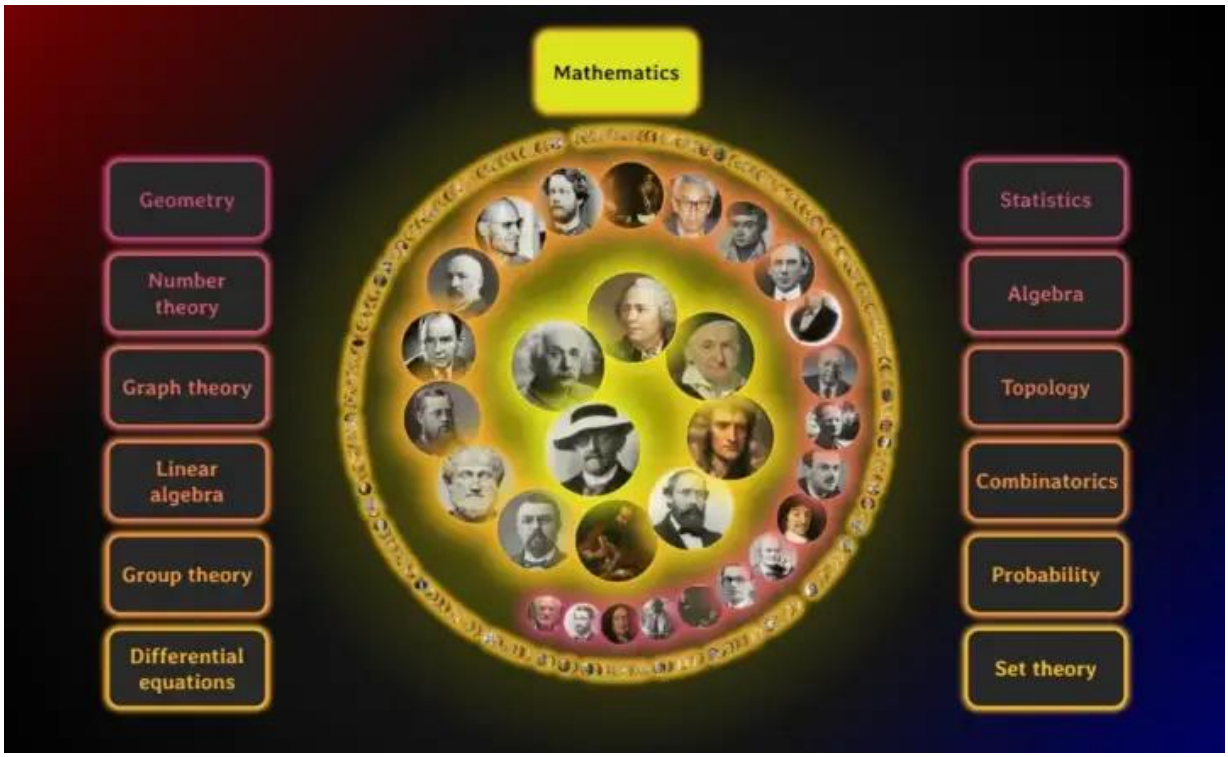
The Greatest Mathematicians
A Data Visualisation

Our modern society is built upon mathematics. And mathematics is built upon the achievements of thousands of outstanding individuals. Explorers, creators, problem solvers… they each pushed the boundaries of knowledge and invented new methods of thinking about the world.

Who was the greatest mathematician of all time? How would you even decide? Is it the most creative genius? The most prolific? The one who solved the most important problem?

This article presents a visualisation of the greatest mathematicians, ranked according to how much **influence** they have on mathematics today.



The main sections are:

- The interactive data visualisation.
- Comments on the overall top 10 ranked mathematicians.
- A list of top-ranked mathematicians by field.
- A discussion of the ranking methodology.

Please visit the web link to get real inter-active experience:

https://www.cantorsparadise.com/the-greatest-mathematicians-a-data-visualisation-716a629ff6a2

You can tap a mathematician's image to show their name and more information, or tap a topic button to show the top-ranked mathematicians for that topic.

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SAPAVIVA

Welcome to sapaviva.com:

Your definitive guide to the world's preeminent scientists (and mathematicians). My desire to study, evaluate and comparatively analyze remarkable works of science: from antiquity to the year 2000 AD inspired this publication. The unrestricted scope spanned across vast discoveries in pure and applied sciences. It is the most encompassing, the most researched, as well as the most unbiased rankings of scientists (including mathematicians) from every part of the world. …A product of over 15 years research and in-depth analyses of the most spectacular innovations this world has ever witnessed. My narratives on this website are ample overviews which were concisely structured for your easy reading. The rankings were based on merit. My four evaluative criteria (each of which bears 25% of the assessment points) are: overall ability, versatility, productivity and developmental influences. To peruse any of those scientists, click on the name or on the portrait. You may also use the *more information* button, which is situated directly beneath this paragraph, to access additional details. Your views (whether criticisms or compliments) are welcomed; so, feel free to voice your opinion via the ***Leave A Reply*** contact channel.
Thank you, and do enjoy your visit.

This Web Page contains:

- 100 Greatest Scientists
- 50 Greatest Mathematicians
- 200 Quotes of Valentine Oduenyi

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Earliest mathematicians

Little is known of the earliest mathematics, but the famous *Ishango Bone* from Early Stone-Age Africa has tally marks suggesting arithmetic. The markings include six prime numbers (5, 7, 11, 13, 17, 19) in order, though this is probably coincidence.

Some of the earliest arithmetic was for economic transactions; this informs us not just about ancient math but about the socio-economic structure of the earliest civilizations. Measuring real estate or silver accurately would have been unnecessary in collectivist societies, or if informal gift exchanges were the norm. In addition to developing geometry to measure irregular-shaped plots of land, both ancient Sumerians and ancient Harappa developed small weights. The Sumerian system was based on grains of barley; a common weight was about 420 milligrams, which is 1/20 of the shekel used in southern Mesopotamia or allegedly the weight of nine grains of barley. Dr, William Hafford made an interesting discovery: He found a set of agate weights weighing 1, 2, 3, 5 or 8 times that 9-grain unit. The Fibonacci numbers! Hafford shows that the Fibonacci numbers allow a convenient weighing procedure. Tiny agate weights of 1/3 or 2/3 of that 9-grain unit have also been found. Such weights could only have been used for weighing very precious things: silver and gold, or perhaps spice. (No such tiny Harappan weight has turned up. Their weights followed a binary system: 1, 2, 4, 8, 16, 32, 64 -- convenient for a different weighing procedure.)

The advanced artifacts and architectures of Egypt's Old Kingdom and the Indus-Harrapa civilization imply strong mathematical skill, but the first written evidence of advanced arithmetic dates from Sumeria, where 4500-year old clay tablets show multiplication and division problems; the first abacus may be about this old. By 3600 years ago,

Mesopotamian tablets show tables of squares, cubes, reciprocals, and even logarithms and trig functions, using a primitive place-value system (in base 60, not 10). Babylonians were certainly familiar with the Pythagorean Theorem; the Plimpton 322 artifact from about the time of Hammurabi the Great contained a table of Pythagorean triples (possibly used to assist land mensuration); some of these triplets are so large (one is 12709, 13500, 18541 -- generated by p=125, q=54) that they almost certainly knew the Pythagorean generating formula. Ancient Mesopotamians also had solutions to quadratic equations and even cubic equations (though they didn't have a general solution for these); they eventually developed methods to estimate terms for compound interest. The Greeks borrowed from Babylonian mathematics, which was the most advanced of any before the Greeks; but there is no ancient Babylonian mathematician whose name is known.

Also at least 3600 years ago, the Egyptian scribe Ahmes produced a famous manuscript (now called the *Rhind Papyrus*), itself a copy of a late Middle Kingdom text. It showed simple algebra methods and included a table giving optimal expressions using Egyptian fractions. (Today, Egyptian fractions lead to challenging number theory problems with no practical applications, but they may have had practical value for the Egyptians. To divide 17 grain bushels among 21 workers, the equation 17/21 = 1/2 + 1/6 + 1/7 has practical value, especially when compared with the "greedy" decomposition 17/21 = 1/2 + 1/4 + 1/17 + 1/1428.)

The Pyramids demonstrate that Egyptians were adept at geometry, though little written evidence survives. Babylon was much more advanced than Egypt at arithmetic and algebra; this was probably due, at least in part, to their place-value system. But although their base-

60 system survives (e.g. in the division of hours and degrees into minutes and seconds) the Babylonian notation, which used the equivalent of IIIIII XXXXXIIIIIII XXXXIII to denote 417+43/60, was unwieldy compared to the "ten digits of the Hindus." (In 2016 historians were surprised to decode ancient Babylonian texts and find very sophisticated astronomical calculations of Jupiter's orbit.)

The Egyptians used the approximation $\pi \approx (4/3)^4$ (derived from the idea that a circle of diameter 9 has about the same area as a square of side 8). Although the ancient Hindu mathematician Apastambha had achieved a good approximation for $\sqrt{2}$, and the ancient Babylonians an ever better $\sqrt{2}$, neither of these ancient cultures achieved a π approximation as good as Egypt's, or better than $\pi \approx 25/8$, until the Alexandrian era.

The sudden blossoming of math in the Iron Ages of India and Greece owes much to the ancient mathematics of Egypt and Babylonia.

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Early Vedic mathematicians

The greatest mathematics before the Golden Age of Greece may have been in India's early Vedic (Hindu) civilization. The Vedics understood relationships between geometry and arithmetic, developed astronomy, astrology, calendars, and used mathematical forms in some religious rituals.

The earliest mathematician to whom definite teachings can be ascribed was Lagadha, who apparently lived about 1300 BC and used geometry and elementary trigonometry for his astronomy. Baudhayana lived about 800 BC and also wrote on algebra and geometry; Yajnavalkya lived about the same time and is credited with the then-best approximation to π. Apastambha did work summarized

below; other early Vedic mathematicians solved quadratic and simultaneous equations.

Other early cultures also developed some mathematics. The ancient Mayans apparently had a place-value system with zero before the Hindus did; Aztec architecture implies practical geometry skills. Ancient China certainly developed mathematics; in fact the first known proof of the Pythagorean Theorem is found in a Chinese book (*Zhoubi Suanjing*) which *might* have been written about 1000 BC.

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Published Works of Ramanujan

Published works of Srinivasa Ramanujan

1. Some properties of Bernoulli's numbers
   Journal of the Indian Mathematical Society, III, 1911, 219 - 234
2. On question 330 of Professor Sanjana
   Journal of the Indian Mathematical Society, IV, 1912, 59 - 61
3. Note on a set of simultaneous equations
   Journal of the Indian Mathematical Society, IV, 1912, 94 - 96
4. Irregular numbers
   Journal of the Indian Mathematical Society, V, 1913, 105 - 106
5. Squaring the circle
   Journal of the Indian Mathematical Society, V, 1913, 132

6. Modular equations and approximations to ππ
   Quarterly Journal of Mathematics, XLV, 1914, 350 - 372
7. On the integral ∫0xtan-1ttdt∫0xtan−1⁡ttdt
   Journal of the Indian Mathematical Society, VII,1915, 93 - 96
8. On the number of divisors of a number
   Journal of the Indian Mathematical Society, VII, 1915, 131 - 133
9. On the sum of the square roots of the first nn natural numbers
   Journal of the Indian Mathematical Society, VII, 1915, 173 - 175
10. On the product ∏n=∞n=0[1+(xa+nd)3]∏n=0n=∞[1+(xa+nd)3]
    Journal of the Indian Mathematical Society, VII, 1915, 209 - 211

11. Some definite integrals
    Messenger of Mathematics, XLIV, 1915, 10 - 18
12. Some definite integrals connected with Gauss's sums
    Messenger of Mathematics, XLIV, 1915, 75 - 85
13. Summation of a certain series
    Messenger of Mathematics, XLIV, 1915, 157 - 160
14. New expressions for Riemann's functions ξ(s)ξ(s) and Ξ(t)Ξ(t)
    Quarterly Journal of Mathematics, XLVI, 1915, 253 - 260
15. Highly composite numbers
    Proceedings of the London Mathematical Society, 2, XIV, 1915, 347 - 409

16. On certain infinite series
Messenger of Mathematics, XLV, 1916, 11 – 15
17. Some formulæ in the analytic theory of numbers
Messenger of Mathematics, XLV, 1916, 81 – 84
18. On certain arithmetical functions
Transactions of the Cambridge Philosophical Society, XXII, No.9, 1916, 159 – 184
19. A series for Euler's constant γγ
Messenger of Mathematics, XLVI, 1917, 73 – 80

20. On the expression of a number in the form ax2+by2+cz2+du2ax2+by2+cz2+du2
Proceedings of the Cambridge Philosophical Society, XIX, 1917, 11 – 21
21. On certain trigonometrical sums and their applications in the theory of numbers
Transactions of the Cambridge Philosophical Society, XXII, No.13, 1918, 259 – 276
22. Some definite integrals
Proceedings of the London Mathematical Society, 2, XVII, 1918,Records for 17 Jan. 1918
23. Some definite integrals
Journal of the Indian Mathematical Society, XI, 1919, 81 – 87
24. A proof of Bertrand's postulate
Journal of the Indian Mathematical Society, XI, 1919, 181 – 182
25. Some properties of p(n)p(n), the number of partitions of nn
Proceedings of the Cambridge Philosophical Society, XIX, 1919, 207 – 210

26. Proof of certain identities in combinatory analysis
Proceedings of the Cambridge Philosophical Society, XIX, 1919,214 – 216
27. A class of definite integrals
Quarterly Journal of Mathematics, XLVIII, 1920, 294 – 310
28. Congruence properties of partitions
Proceedings of the London Mathematical Society, 2, XVIII, 1920, Records for 13 March 1919
29. Algebraic relations between certain infinite products
Proceedings of the London Mathematical Society, 2, XVIII, 1920, Records for 13 March 1919
30. Congruence properties of partitions
Mathematische Zeitschrift, IX, 1921, 147 – 153

31. Une formule asymptotique pour le nombre des partitions de nn
(written in collaboration with G. H. Hardy)
Comptes Rendus, 2 Jan. 1917

32. Proof that almost all numbers nn are composed of about log log nn prime factors
(written in collaboration with G. H. Hardy)
Proceedings of the London Mathematical Society, 2, XVI,1917, Records for 14 Dec. 1916

33. Asymptotic formulæ in combinatory analysis
(written in collaboration with G. H. Hardy)
Proceedings of the London Mathematical Society, 2, XVI, 1917, Records for 1 March 1917

34. Asymptotic formulæ for the distribution of integers of various types
(written in collaboration with G. H. Hardy)
Proceedings of the London Mathematical Society, 2, XVI, 1917, 112 – 132

35. The normal number of prime factors of a number nn
(written in collaboration with G. H. Hardy)
Quarterly Journal of Mathematics, XLVIII, 1917, 76 – 92

36. Asymptotic formulæ in combinatory analysis
(written in collaboration with G. H. Hardy)
Proceedings of the London Mathematical Society, 2, XVII, 1918, 75 – 115

37. On the coefficients in the expansions of certain modular functions
(written in collaboration with G. H. Hardy)
Proceedings of the Royal Society, A, XCV, 1919, 144 – 155

Source:

http://ramanujan.sirinudi.org/html/published_papers.html

Ramanujan

இராமானுஜன்

Mathematician Srinivasa Ramanujan Biographical Movie

Watch Tamil Full Movie Ramanujan (2014) Biographical film based on the life of Indian Mathematician Srinivasa Ramanujan. [with English sub-titles]

Starring : Abhinay Vaddi, Suhasini Maniratnam, Bhama, Kevin Mc Gowan, Nizhalgal Ravi, Richard Walsh, Abbas.

Director : Gnana Rajasekaran

Producer : Srivatsan Nadathur, Sushant Desai, Sharanyan Nadathur, Sindhu Rajasekaran

Duration: 2h 35m

https://www.youtube.com/watch?v=zx6QsOdA4cQ&ab_channel=TamilMovieplex

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Award:

Ramanujan won the **Tamil Nadu State Film Award for Best Film** of 2013.

Background

Set in the early 1900s, the film traces the life of the prodigious math genius Srinivasa Ramanujan from the time he was a young Tamil Brahmin to his years in England, where he attended Cambridge University during World War I. The film follows his relationships with his mother Komalatammal, his wife Janaki, and his collaborator Professor G. H. Hardy. The film also showcases how Indian society viewed a mathematician of such great stature.

Production

Gnana Rajasekaran looked for an actor who would physically resemble Ramanujan and signed Abhinay Vaddi, who is the grandson of veteran Tamil actor Gemini Ganesan, for the main role. Michael Lieber was signed to play Edensor Littlewood. Lieber confessed he had no knowledge of Ramanujan, when he signed the film, and went as far as to meet Béla Bollobás, who worked with Littlewood personally, for his research work. He found it difficult to speak the Tamil lines, but appreciated it later on, "I would be lying if I said that the task of learning Tamil was not daunting at first, but once I got the hang of it, I was able to appreciate what a beautiful language it is. There were many different ways of learning the Tamil lines; some used audio tapes, prompting, flash cards, or word boards. I learnt the meaning of the words and memorised chunks of dialogue."[1] Similarly, Kevin McGowan, who was signed for the role of G. H. Hardy, was unfamiliar with the story of Ramanujan. The film also stars Bhama, Suhasini Maniratnam, Abbas and Richard Walsh amongst others. Sunny Joseph, best known for his high-profile Malayalam works with directors Adoor Gopalakrishnan and Shaji N. Karun, was signed as cinematographer.

Shooting:

The film has been shot in the five main locations of Ramanujan's life, Kumbakonam, Namakkal, Chennai, London and Cambridge. The first two schedules were shot in India while the third was done in England, where they took the permission of Cambridge University to shoot.[10] The task to create a script in multiple languages was described by Sindhu Rajasekaran as "quite a chore. Roxane de Rouen and I are working with Gnana Rajasekaran, the director of 'Ramanujan,' to make the characters speak words that make them real. No, you wouldn't find roadside urchins who speak in Queen's English here; people would speak what comes to them naturally: Tamil, English, Indian English, even Tamenglish. What a delight it is to live in this world where languages are not borders, but an element to experiment with..."

Music:

The film's soundtrack and score were composed by Ramesh Vinayakam. Vinayakam called it an honour to have composed music for the film and "an opportunity for me to travel on a new path" and added that he was given the required creative freedom by the director and producer. He said that he had to go classical for one part of the film and into European sounds for another segment. No modern instruments, but old world instruments were used in order to bring the period alive, while four orchestral pieces were recorded in Germany, where the composer worked with the Stuttgart-based GermanPops Orchestra that has worked on contemporary and classical styles. Being a film about a mathematician, the songs were related to mathematical terms as well. Poet Vaali had written a number, "Narayana Narayana", which was said to "incorporate the idea of infinity and nothingness", while another number, written by Thirumalisai Alvar thousands of years ago, "stresses that numbers are absolute". One of the instrumental tracks, "One to Zero" was described as a "musical metaphor" as it was based on the numbers, with Vinayakam explaining, "One is represented by one note, two by two notes and so on, while zero is represented by a rest. After the initial exposition, at any point of time, three of these layers criss-cross each other from a distance of one beat from each other"."Narayana" was sung by Vani Jayaram, who said that she was initially surprised that she was given a song whose pitch (shruthi) was less than what she usually sings in but learned that, in the days of Ramanujan, people only sang in that pitch.

The soundtrack album of *Ramanujan* was released at the Suryan FM radio station in Chennai on 13 June 2014. The album, which features eight tracks, including four songs and four instrumentals, was praised by critics. Indiaglitz in its review wrote, "Ramesh Vinayakam was always there and thereabouts with his wonderful albums in the past. With this album he has struck the right chords towards his name once again. This must be the break he's been looking for years". musicaloud.com gave it a score of 9 out of 10 and wrote, "Ramesh Vinayakam expertly draws from Carnatic and Western classical styles to produce one of the finest period film soundtracks ever". Behindwoods.com gave it 3 stars out of 5 and wrote, "*Ramanujan* enthralls the listener by transporting them to a bygone era of classical music". Noted Malayalam film composer M. Jayachandran, praised Vinayakam for his music in *Ramanujan*. The album was also named by Deccan Music and Milliblog as the best Tamil music album of the year 2014.

Release:

The official trailer of the film was released on 16 June 2014. On 9 July 2014, the producers of Ramanujan arranged a special screening at the Rashtrapati Bhavan, receiving an invitation from the president Pranab Mukherjee.[23] The film was simultaneously released in India and United Kingdom in Tamil and English languages on 11 July 2014.

Actors:

- Abhinay Vaddi as Srinivasa Ramanujan
- Suhasini Maniratnam as Komalatammal (Srinivasa Ramanujan's Mother)
- Bhama as Janaki (Srinivasa Ramanujan's Wife)
- Kevin McGowan as G. H. Hardy
- Abbas as Prasanta Chandra Mahalanobis
- Anmol as Young Srinivasa Ramanujan
- Michael Lieber as John Edensor Littlewood
- Richard Walsh as Francis Spring
- Sharath Babu as Diwan Bahadur R. Ramachandra Rao I. C. S.
- Radha Ravi as Prof. Singaravelu Mudaliar
- Madhan Bob as Prof. Krishna Shastri
- Y. G. Mahendran as S. Narayana Iyer
- Manobala as Krishna Rao
- Nizhalgal Ravi as Srinivasa Raghavan
- Satish Kumar as Anandhu
- Thalaivasal Vijay as Sathiyapriya Rayar
- Manibharathi as Krishnan
- Delhi Ganesh
- Raja Krishnamoorthy as Seshu Iyer
- T. P. Gajendran as T. Namberumal Chetty
- Mohan V. Ram
- Cloudia Swann as Ms. Bourne
- Mike Parish as Doctor Charles
- Harsh Naik as Chatterjee
- Lizzie Bourne as Ms. Gertrude Hardy

Reviews:

The film received mixed reviews from critics, who generally praised the acting but criticized the writing. The *Deccan Chronicle* called *Ramanujan* "a brilliant piece on canvas with edifying moments and relevance to modern age" and went on to add that it was "not to be missed", giving it 3.5/5 stars. S. Saraswathi of *Rediff* wrote, "*Ramanujan* is a brilliant film, a must watch" and gave the film 4/5.[26] Gautaman

Bhaskaran of *The Hindustan Times* gave the film 3/5 stars and wrote, "The movie is a poignant look at the way a prodigy struggled and suffered in a penurious family, a mastermind whose mathematical wizardry invited ridicule and revulsion in far lesser mortals. Rajasekaran, who also scripted the film, takes us through a linear narrative to tell us about the intelligence of boy Ramanujan as he completely foxes his school-master with a little insight into the importance of zero, and later about his frustration when he hits a wall in his quest to sink into, and shine, with numbers". *Sify* wrote, "Making biopics is indeed very challenging and the director has been successful to a very large extent to bring out each and every character. Gnana Rajasekaran has done a well-researched biopic on Ramanujam...it is a film that is definitely worth viewing". *IANS* gave it 3/5 and wrote, "Gnana Rajasekaran certainly knows the art and succeeds narrating an inspiring tale, but his work doesn't resonate deep within. This is so because the director merely recreates several important episodes from Ramanujan's life on the screen while ignoring the need to build a screenplay to keep the viewers hooked". Bharath Vijayakumar of *Moviecrow* rated 3/5 stars and said, "Ramanujan is a noble effort and a fascinating insight about the life and times of this Maths Wizard who lived all his life in unison with his true love.".

In contrast, *The New Indian Express* wrote, "The director's persistent effort to bring on celluloid lives of eminent people is laudable. But a movie is not only about the theme, but also about how it is presented on screen. And the presentation of the life and journey of the mathematical genius is disappointing and uninspiring". Baradwaj Rangan of *The Hindu* wrote, "The film runs nearly three hours and it's puzzling why it needed to. There appears to have been no effort to streamline the events of Ramanujan's life. The writing, too, fails to make *Ramanujan* interesting", going on to add "The great man certainly deserved a better movie". M. Suganth of the *Times of India* gave 2.5/5 and wrote, "For a film that is about a man with astounding talent, the filmmaking is largely unimaginative. The staging is somewhat old-fashioned (read dated), the pacing staid and the film often slips into the kind of melodrama that you nowadays find in TV serials.". *Daily India* gave 2.5/5 and stated, "In overall, *Ramanujan* is a cleanly made and it's a very rare kind of movie in biopics genre. Editing and cinematography is worth praising. Abhinay Vaddi, the grandson of veteran Tamil film actor Gemini Ganesan done a good job and made justice to their roles.". *Indiaglitz* gave 2.25/5 and wrote, "An honest attempt to drive through the life of a genius gets stuck at lot of bumpers.".

(😂)(😂)(😂)☹(😂)

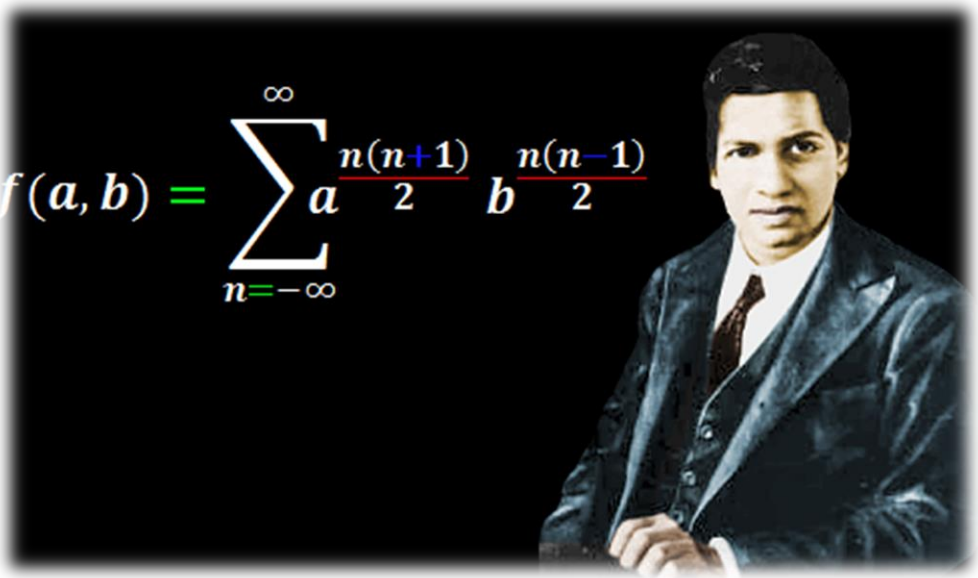
Source:

https://en.wikipedia.org/wiki/Ramanujan_%28film%29

Biography

[Srinivasa Ramanujan]

[22 12 1887 ::: 26 04 1920]



Srinivasa Ramanujan was a largely self-taught pure mathematician. Hindered by poverty and ill-health, his highly original work has considerably enriched number theory. More recently his discoveries have been applied to physics, where his theta function lies at the heart of string theory.

Beginnings

Srinivasa Ramanujan was born on December 22, 1887 in the town of Erode, in Tamil Nadu, in the south east of India. His father was K. Srinivasa Iyengar, an accounting clerk for a clothing merchant. His mother was Komalatammal, who earned a small amount of money each month as a singer at the local temple.

His family were Brahmins, the Hindu caste of priests and scholars. His mother ensured the boy was in tune with Brahmin traditions and culture. Although his family were high caste, they were very poor.

Ramanujan's parents moved around a lot, and he attended a variety of different elementary schools.

Early Mathematics

At age 10, Ramanujan was the top student in his district and he started high school at the Kumbakonam Town High School. Looking at the mathematics books in his school's library, he quickly found his vocation. By age 12, he had begun serious self-study of mathematics, working through cubic equations and arithmetic and geometric series. He invented his own method of solving quartic equations.

As Ramanujan's mathematical knowledge developed, his main source of inspiration and expertise became *Synopsis of elementary results in pure mathematics* by George S. Carr. This book presented a very large number of mathematical results – over 4000 theorems – but generally showed little working, cramming into its pages as many results as possible.

2478 $\int_0^{\pi} \cos^p x \cos nx\, dx = C\left(p, \frac{p-n}{2}\right)\frac{\pi}{2^p}$,

when p and n are either *both odd* or *both even*, and n is not greater than p.

Entry 2478 from Carr's *Synopsis of elementary results in pure mathematics*

With little other guidance, Ramanujan came to believe this was how mathematics was done, so he himself learned to show little working. Also, he could afford only a small amount of paper, doing most of his work on slate with chalk, transferring a minimal amount of his working and his results to paper.

His memory for mathematical formulas and constants seems to have been boundless: he amazed classmates with his ability to recite the values of irrational numbers like π, e, and √2 to as many decimal places as they asked for.

An Apparently Bright Future Fizzles Out

In 1904, Ramanujan left high school; his future looked promising: he had won the school's mathematics prize and, more importantly, a scholarship allowing him to study at the Government Arts College in the town of Kumbakonam.

Obsessed with mathematics, Ramanujan failed his non-mathematical exams and lost his scholarship. In 1905, he travelled to Madras and enrolled at Pachaiyappa's College, but again failed his non-mathematical exams.

The Discovery of Ramanujan as a Mathematician of Genius

The Hungry Years

At the beginning of 1907, at age 19, with minimal funds and a stomach all too often groaning with hunger, Ramanujan continued on the path he had chosen: total devotion to mathematics. The mathematics he was doing was highly original and very advanced.

Even though (or some might say because) he had very little formal mathematical education he was able to discover new theorems. He also independently discovered results originally discovered by some of the greatest mathematicians in history, such as Carl Friedrich Gauss and Leonhard Euler.

Ill-health was Ramanujan's constant companion – as it would be for much of his short life.

By 1910, he realized he must find work to stay alive. In the city of Madras he found some students who needed mathematics tutoring and he also walked around the city offering to do accounting work for businesses.

And then a piece of luck came his way. Ramanujan tried to find work at the government revenue department, and there he met an official whose name was Ramaswamy Aiyer. Ramanujan did not have a resume to show Ramaswamy Aiyer; all he had were his notebooks – the results of his mathematical work.

Ramanujan's good fortune was that Ramaswamy Aiyer was a mathematician. He had only recently founded the Indian Mathematical Society, and his jaw dropped when he saw Ramanujan's work.

"I was struck by the extraordinary mathematical results contained in it. I had no mind to smother his genius by an appointment in the lowest rungs of the revenue department."

V. RAMASWAMY AIYER, 1871 – 1936
Mathematician

Things Begin to Look Up

Ramaswamy Aiyer contacted the secretary of the Indian Mathematical Society, R. Ramachandra Rao, suggesting he provide financial support for Ramanujan. At first Rao resisted the idea, believing Ramanujan was simply copying the work of earlier great mathematicians. A meeting with Ramanujan, however, convinced Rao that he

was dealing with a genuine mathematical genius. He agreed to provide support for Ramanujan, and Ramaswamy Aiyer began publishing Ramanujan's work in the *Journal of the Indian Mathematical Society*.

Ramanujan's work, however, was hard to understand. The style he had adopted as a schoolboy, after digesting George S. Carr's book, contributed to the problem. His mathematics often left too few clues to allow anyone who wasn't also a mathematical genius to see how he obtained his results.

In March 1912, his financial position improved when he got a job as an accounting clerk with the Madras Port Trust.

There he was encouraged to do mathematics at work after finishing his daily tasks by the port's Chief Accountant, S. Narayana Iyer, who was treasurer of the Indian Mathematical Society, and by Sir Francis Spring, an engineer, who was Chairman of the Madras Port Trust.

Francis Spring began pressing for Ramanujan's mathematical work to be supported by the government and for him to be appointed to a research position at one of the great British universities.

A Crank or a Genius?

Ramanujan and his supporters contacted a number of British professors, but only one was receptive – an eminent pure mathematician at the University of Cambridge – Godfrey Harold Hardy, known to everyone as G. H. Hardy, who received a letter from Ramanujan in January 1913. By this time, Ramanujan had reached the age of 25.

Professor Hardy puzzled over the nine pages of mathematical notes Ramanujan had sent. They seemed rather incredible. Could it be that one of his colleagues was playing a trick on him?

Hardy reviewed the papers with J. E. Littlewood, another eminent Cambridge mathematician, telling Littlewood they had been written by either a crank or a genius, but he wasn't quite sure which. After spending two and a half hours poring over the outlandishly original work, the mathematicians came to a conclusion. They were looking at the papers of a mathematical genius:

"I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written by a mathematician of the highest class. They must be true because, if they were not true, no one would have the imagination to invent them."

G. H. HARDY, 1877 – 1947
Mathematician

Hardy was eager for Ramanujan to move to Cambridge, but in accordance with his Brahmin beliefs, Ramanujan refused to travel overseas. Instead, an arrangement was made to fund two years of work at the University of Madras. During this time, Ramanujan's mother had a dream in which the goddess Namagiri told her she should give her son permission to go to Cambridge, and this she did. Her decision led to several very heated quarrels with other devout family members.

Ramanujan at Cambridge

Ramanujan arrived in Cambridge in April 1914, three months before the outbreak of World War 1. Within days he had begun work with Hardy and Littlewood. Two years later, he was awarded the equivalent of a Ph.D. for his work – a mere formality.

Srinivasa Ramanujan at Cambridge

Ramanujan's prodigious mathematical output amazed Hardy and Littlewood.

The notebooks he brought from India were filled with thousands of identities, equations, and theorems he discovered for himself in the years 1903 – 1914.

Some had been discovered by earlier mathematicians; some, through inexperience, were mistaken; many were entirely new.

"It was his insight into algebraical formulae, transformations of infinite series, and so forth that was most amazing. On this side most certainly, I have never met his equal, and I can compare him only with Euler or Jacobi."

G. H. HARDY, 1877 – 1947
Mathematician

Explaining Ramanujan's Extraordinary Mathematical Output

Ramanujan had very little formal training in mathematics, and indeed large areas of mathematics were unknown to him. Yet in the areas familiar to him and in which he enjoyed working, his output of new results was phenomenal.

Ramanujan said the Hindu goddess Namagiri – who had appeared in his mother's dream telling her to allow him to go to Cambridge – had appeared in one of his own dreams.

"While asleep, I had an unusual experience. There was a red screen formed by flowing blood, as it were. I was observing it. Suddenly a hand began to write on the screen. I became all attention. That hand wrote a number of elliptic integrals. They stuck to my mind. As soon as I woke up, I committed them to writing."

SRINIVASA RAMANUJAN, 1887 – 1920
Mathematician

According to Hardy, Ramanujan's ideas were:

"… arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account."

G. H. HARDY, 1877 – 1947
Mathematician

It is possible that Ramanujan's brain was wired differently from most mathematicians.

He seems to have had a personal window through which some problems in number theory appeared with a clarity denied to most people in the field. Results they fought for through days of arduous thought seemed obvious to Ramanujan.

Professor Bruce Berndt is an analytic number theorist who, since 1977, has spent decades researching Ramanujan's theorems. He has published several books about them, establishing that the great majority are correct. He was told an interesting story by the great Hungarian mathematician Paul Erdős about something G. H. Hardy had once said to him:

"Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100. Hardy gave himself a score of 25, Littlewood 30, Hilbert 80 and Ramanujan 100."

PAUL ERDŐS, 1913 – 1996
Mathematician

Given that David Hilbert is regarded by many as the greatest mathematician of the early twentieth century, and Hardy and Littlewood were immensely influential mathematicians, it is fascinating to see how exceptional Hardy thought Ramanujan's raw mathematical ability was.

Number Theory and String Theory

In 1918 Ramanujan became the first Indian mathematician to be elected a Fellow of the British Royal Society:

"Distinguished as a pure mathematician particularly for his investigation in elliptic functions and the theory of numbers."

In his short lifetime he produced almost 4000 proofs, identities, conjectures, and equations in pure mathematics.

His theta function lies at the heart of string theory in physics.

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$$

The Ramanujan theta function.

"… each of the 24 modes in the Ramanujan function corresponds to a physical vibration of a string. Whenever the string executes its complex motions in space-time by splitting and recombining, a large number of highly sophisticated mathematical identities must be satisfied. These are precisely the mathematical identities discovered by Ramanujan."

MICHIO KAKU, BORN 1947
Theoretical Physicist

Some Personal Details and the End

In July 1909, Ramanujan married S. Janaki Ammal, who was then just 10 years old. The marriage had been arranged by Ramanujan's mother. The couple began sharing a home in 1912.

When Ramanujan left to study at the University of Cambridge, his wife moved in with Ramanujan's parents. Ramanujan's scholarship was sufficient for his needs in Cambridge and the family's needs in Kumbakonam.

For his first three years in Cambridge, Ramanujan was very happy. His health, however, had always been rather poor. The winter weather in England, much colder than anything he had ever imagined, made him ill for a time.

In 1917, he was diagnosed with tuberculosis and worryingly low vitamin levels. He spent months being cared for in sanatoriums and nursing homes.

In February 1919, his health seemed to have recovered sufficiently for him to return to India, but sadly he lived for only one more year.

Srinivasa Ramanujan died aged 32 in Madras on April 26, 1920. His death was most likely caused by hepatic amoebiasis caused by liver parasites common in Madras. His body was cremated.

Sadly, some of Ramanujan's Brahmin relatives refused to attend his funeral because he had travelled overseas.

"For my part, it is difficult for me to say what I owe to Ramanujan – his originality has been a constant source of suggestion to me ever since I knew him, and his death is one of the worst blows I have ever had."

G. H. HARDY, 1877 – 1947
Mathematician

"That was the wonderful thing about Ramanujan. He discovered so much, and yet he left so much more in his garden for other people to discover."

FREEMAN DYSON, BORN 1923
Mathematician and Physicist

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Resources:

01] https://www.famousscientists.org/srinivasa-ramanujan/

02] https://byjus.com/free-ias-prep/srinivasa-ramanujan/

03] https://mathshistory.st-andrews.ac.uk/Biographies/Ramanujan/

04] https://www.jagranjosh.com/general-knowledge/srinivasa-ramanujan-biography-1640160796-1

05] https://www.javatpoint.com/srinivas-ramanujan

06] https://www.visionphysics.in/2021/06/Biography-Srinivasa-Ramanujan.html

07] https://www.thoughtco.com/srinivasa-ramanujan-4571004

SRINIVASA RAMANUJAN:
The Mathematician & His Legacy

Each year, December 22nd is celebrated as the National Mathematical Day in India to commemorate the birth anniversary of the self-taught genius – Srinivasa Ramanujan whose work continues to intrigue, inspire and challenge mathematicians around the world. His fertile mind envisioned a world where science and mathematics converge to uncover fundamental principles governing our universe. Nearly a hundred years after his death, mathematics has slowly

developed tools that are unravelling some of the secrets behind his intuitive suppositions. His oeuvre has opened up new vistas of possibilities, new branches of mathematics have sprouted and proliferated and found application in different fields of sciences, implausible in his own times. The Film brings together his life on screen, with period enactments, interviews and animation.

Watch the Film:

https://www.youtube.com/watch?v=uhNGCn_3hmc&ab_channel=IndianDiplomacy

Message a child can learn

from Ramanujan's life

1. Resilience

Srinivasa Ramanujan had to cross many hurdles on his way to success. There was a time when he faced abject poverty and didn't have money to buy a pen and paper. While pursuing higher studies in Chennai, he failed to make a mark in any subject other than Maths and was, therefore, unable to complete his course. He also suffered from poor health. However, none of these difficulties prevented Ramanujan from pursuing his passion relentlessly.

2. Originality and Passion

Ramanujan's life clearly exhibits his passion for exploring the subject rather than just learning what was taught. He pursued his own research on the subject with complete dedication till the very last day of his life. He wrote letters to G H Hardy in 1920, the year he died, even when he was seriously ill, describing complex Mathematical concepts. Such was his love for the subject!

3. Faith and spirituality

"An equation for me has no meaning unless it represents a thought of God." These words by Ramanujan show that he attributed a lot of his early success to faith in God and many of his observers also stated how he exhibited great tolerance and spiritual leanings.

4. Smart thinking

One example of this trait is that since Ramanujan could not afford to buy a lot of paper, he would do most of his work and proofs on his slate, and then transfer just the results to paper.

5. Rich Legacy

Hardy went on to live 27 years after Ramanujan's premature death, but when Hardy was interviewed several years after Ramanujan's death and was asked what his greatest contribution was, he said,

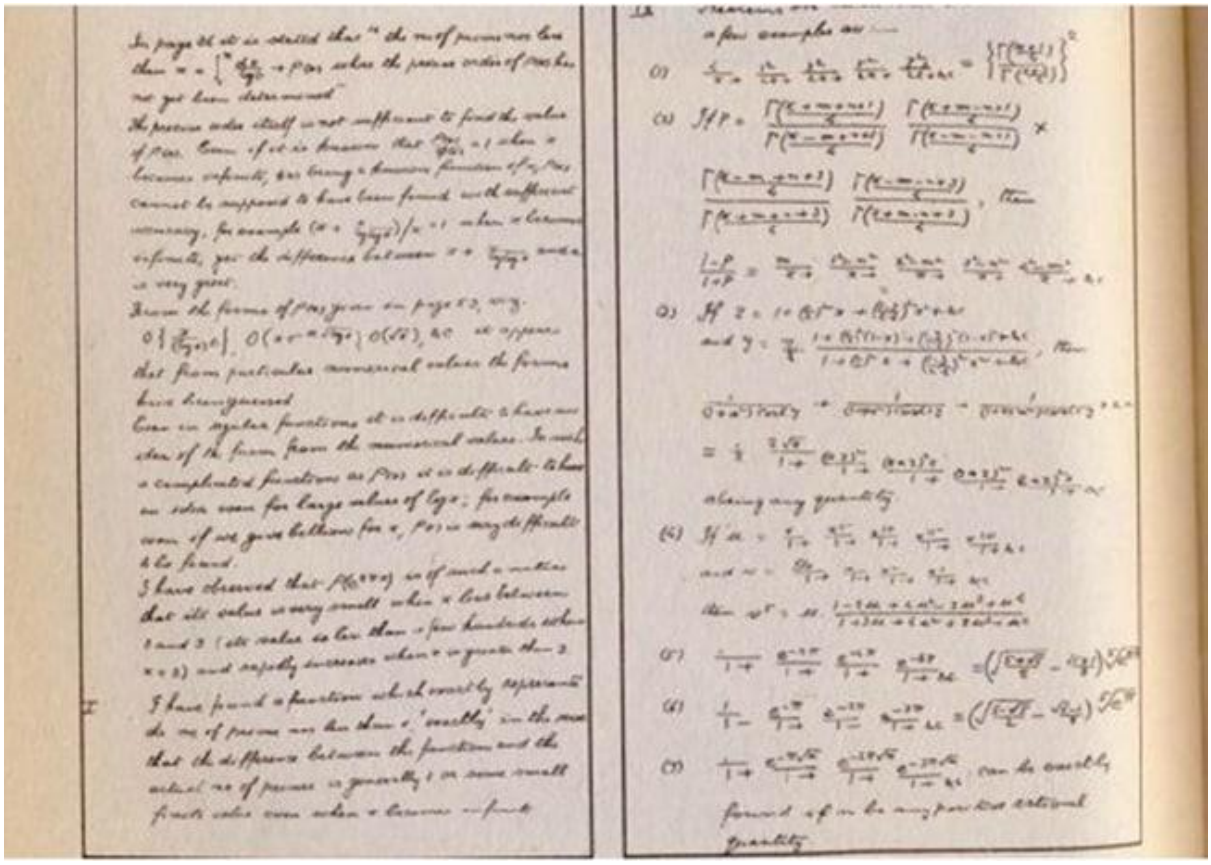
"My discovery of Ramanujan".

Success is not just about getting great grades or making it big in life; it is about pursuing your true passion, come what may. Ramanujan's life story is the perfect example.

Selected Photographs

Ramanujan's House

Ramanujan with his Parents!



Ramanujan's Handwriting

Prof GH Hardy

Ramanujan's Mother

Ramanujan (centre) at Trinity College

Ramanujan's Three Notebooks

1. Ramanujan's mother: Komalatthammal

2. Goddess Namagiri, deity at Namakkal

3. Srinivasa Ramanujan's passport photograph (1919)

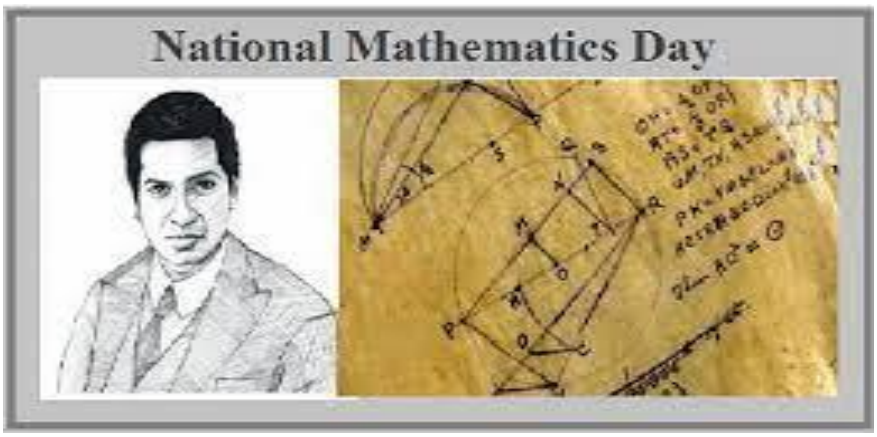
4. Town High School, Kumbakonam, where Ramanujan studied from 1898 to 1903.

Wife Mrs Janaki Ammal

International Monument @
SASTRA Deemed University

Ramanujan with Janaki

In 1909, Ramanujan was married to a 9-year-old child bride, S Janaki Ammal.



S. Janaki Ammal
W/o (Late) Srinivasa Ramanujam
(Mathematical Genius)

S.Ramanujan
1887-1920

45, Muthiah Mudali IInd Street,
Krishnampet, Madras-600 014.
INDIA

Date 17.3.1984

Dear Sir,

I understand from Mr. Richard Askey, Wisconsin, U.S.A., that you have contributed for the sculpture in memory of my late husband Mr. Srinivasa Ramanujan. I am happy over this event.

I thank you very much for your good gesture and wish you success in all your endeavours.

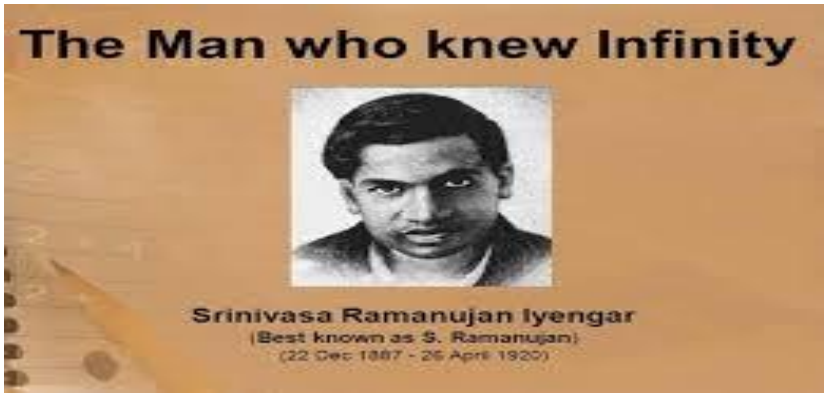
Yours faithfully,

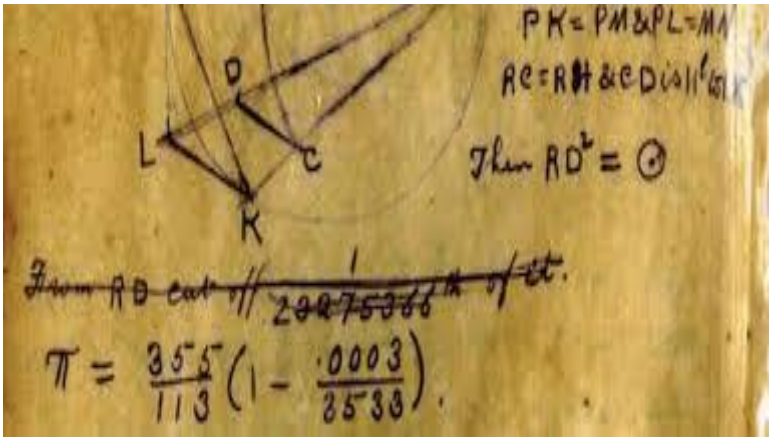
S. Janaki Ammal

S. Janaki Ammal

Ono's father [The mathematician Ken Ono] treasured the letter that Ramanujan's widow sent him in 1984

Making
Mathematics
Meaningful
Remembering
THE GREAT INDIAN MATHEMATICIAN
Srinivasa
Ramanujan
The Man Who Knew Infinity
on National Mathematics Day
22 December 2020
133rd Birth Anniversary


The Man who knew Infinity
Srinivasa Ramanujan Iyengar
(Best known as S. Ramanujan)
(22 Dec 1887 - 26 April 1920)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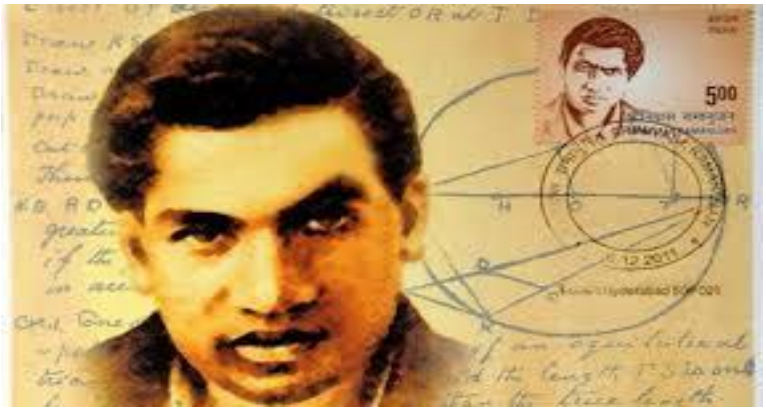
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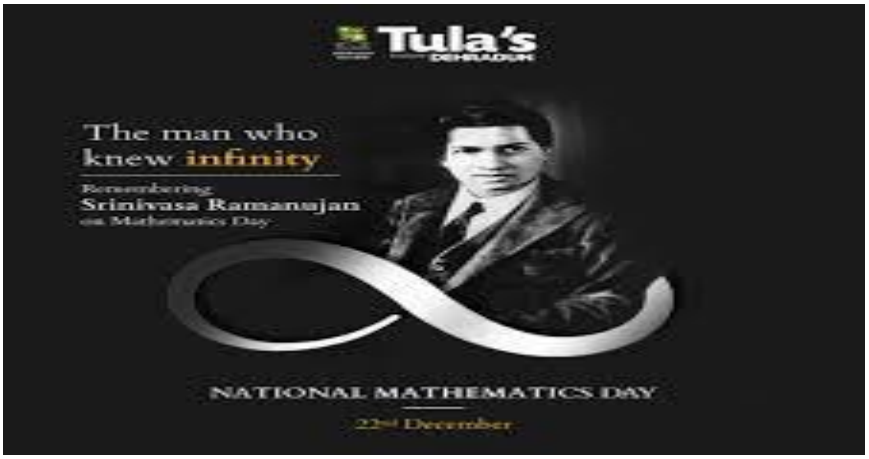
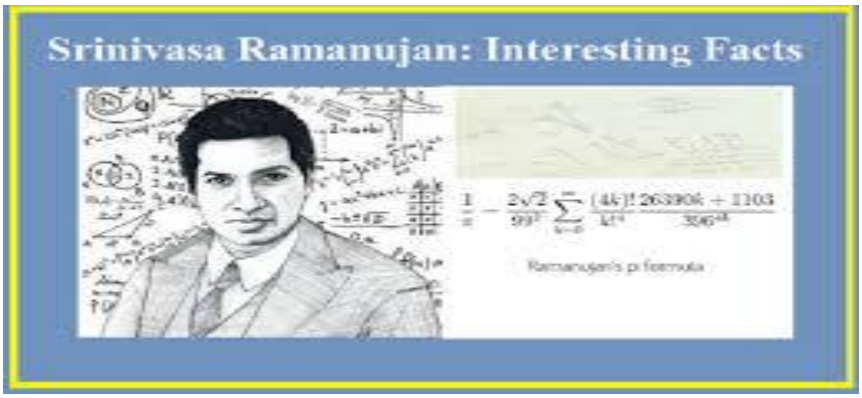
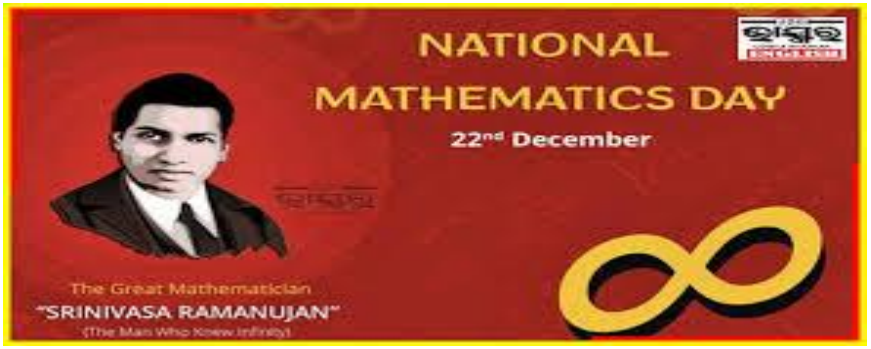
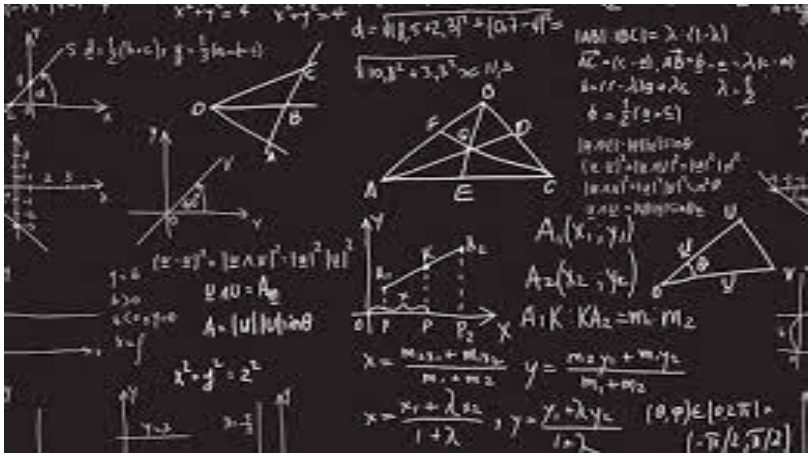
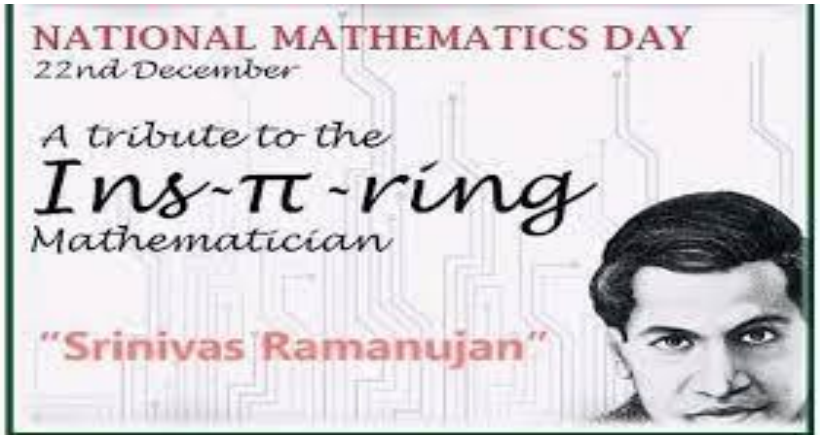

Tula's
DEHRADUN
The man who
knew infinity
Remembering
Srinivasa Ramanujan
on Mathematics Day
NATIONAL MATHEMATICS DAY
22nd December


Srinivasa Ramanujan: Interesting Facts
Ramanujan's pi formula

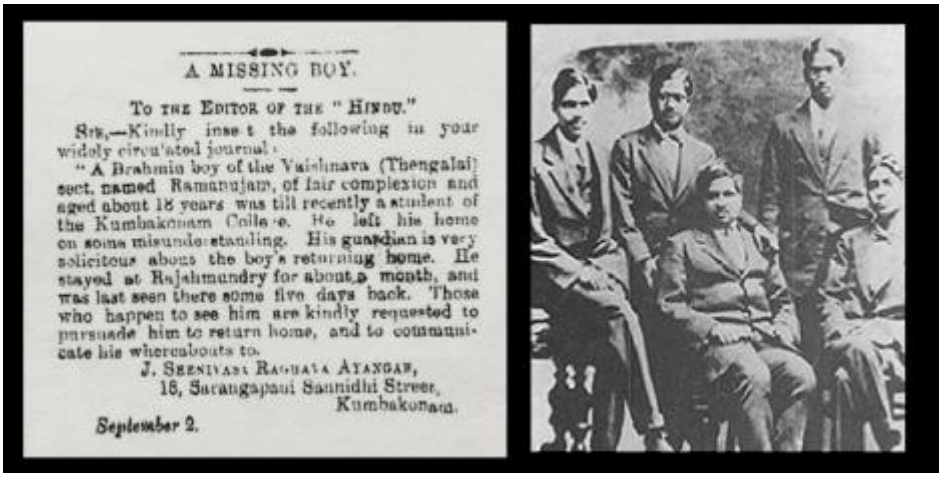

NATIONAL
MATHEMATICS DAY
22nd December
The Great Mathematician
"SRINIVASA RAMANUJAN"
(The Man Who Knew Infinity)

Ramanujan
The Mathemagician




NATIONAL MATHEMATICS DAY
22nd December
A tribute to the
Ins-π-ring
Mathematician
"Srinivas Ramanujan"

SRINIVASA RAMANUJAN
MVLTA ET MIRANDA DE NVMERIS REPPERIT
SINGVLARE INGENIVM AGNOVIT GOTOFRIDVS HARDY
IN ANGLIAM VEXIT STVDIA PROMOVEBAT
HIC PRIMVS INDORVM REGIAE SOCIETATIS SODALIS
HVIVS COLLEGII DEINDE SOCIVS FACTVS
IN PATRIAM AEGER DIV VALETVDINE REGRESSVS
PRAEMATVRAM MORTEM OBIIT A.S. MCMXX



A MISSING BOY.

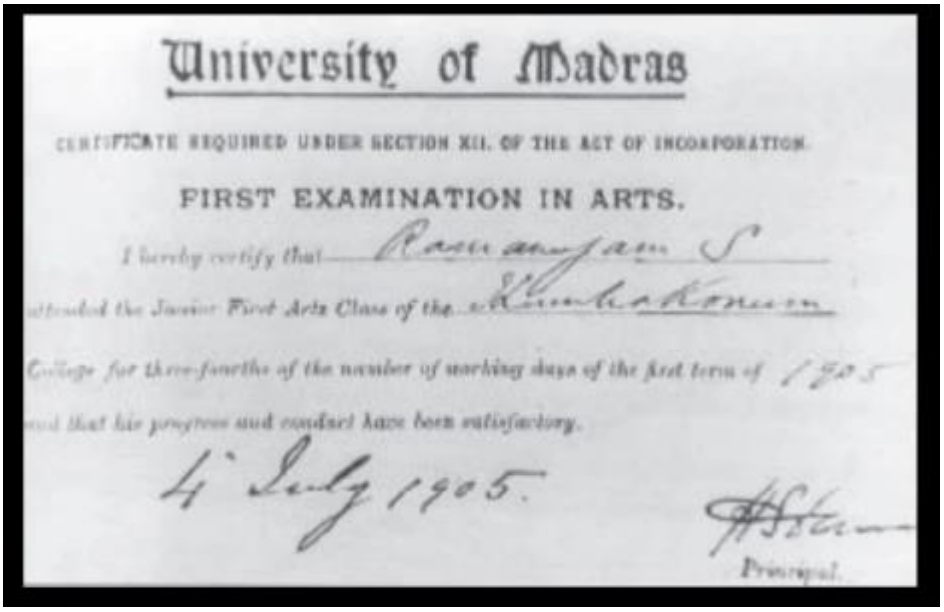
To the Editor of the "Hindu."

Sir,—Kindly insert the following in your widely circulated journal:

"A Brahmin boy of the Vaishnava (Thengalai) sect, named Ramanujam, of fair complexion and aged about 18 years was till recently a student of the Kumbakonam College. He left his home on some misunderstanding. His guardian is very solicitous about the boy's returning home. He stayed at Rajahmundry for about a month, and was last seen there some five days back. Those who happen to see him are kindly requested to persuade him to return home, and to communicate his whereabouts to.

J. Srinivasa Raghava Ayangar,
16, Sarangapani Sannidhi Street,
Kumbakonam.

September 2.



University of Madras

CERTIFICATE REQUIRED UNDER SECTION XII. OF THE ACT OF INCORPORATION.

FIRST EXAMINATION IN ARTS.

I hereby certify that Ramanujam S

attended the Junior First Arts Class of the Kumbakonam

College for three-fourths of the number of working days of the first term of 1905

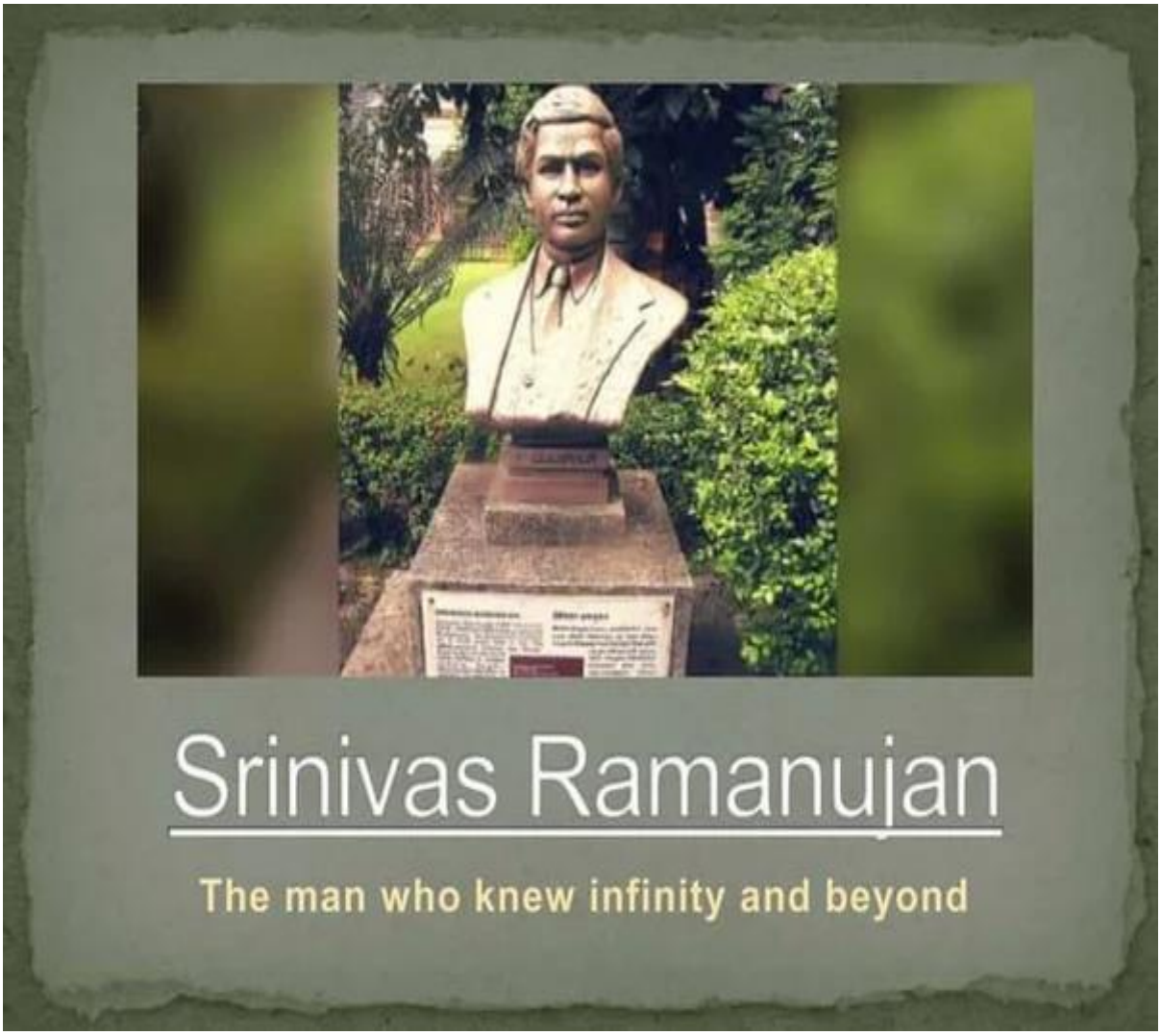
and that his progress and conduct have been satisfactory.

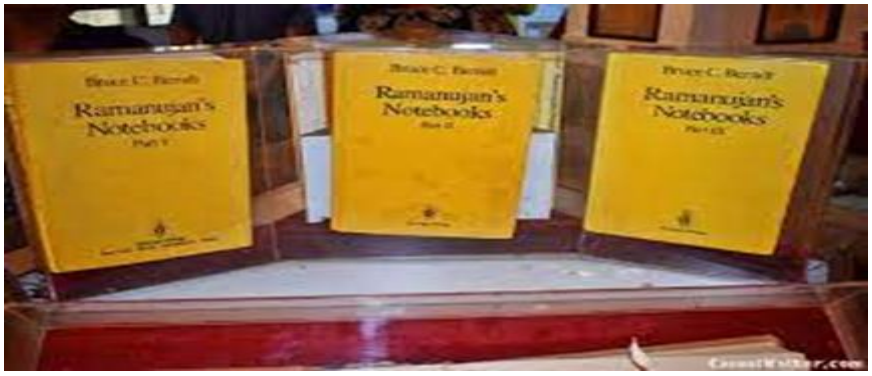
4th July 1905.

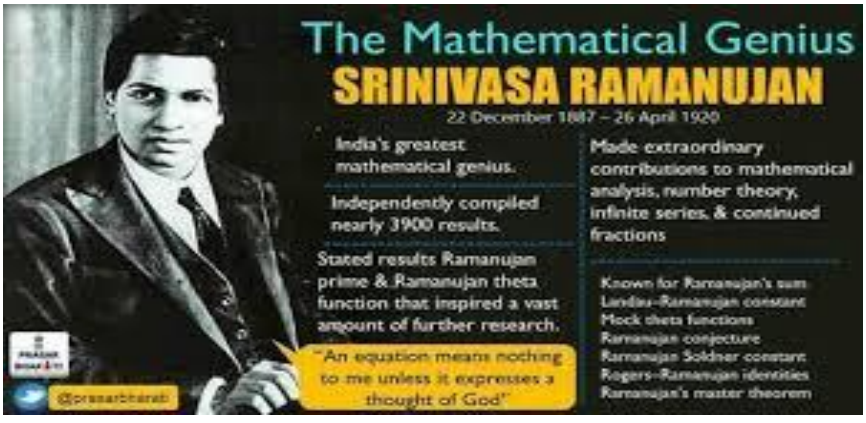
Principal.

And so on March 13, 1919, Ramanujan returned to India—now very famous and respected, but also very ill. Through it all, he continued to do mathematics, writing a notable letter to Hardy about "mock" theta functions on January 12, 1920. He chose to live humbly, and largely ignored what little medicine could do for him. And on April 26, 1920, at the age of 32, and three days after the last entry in his notebook, he died.

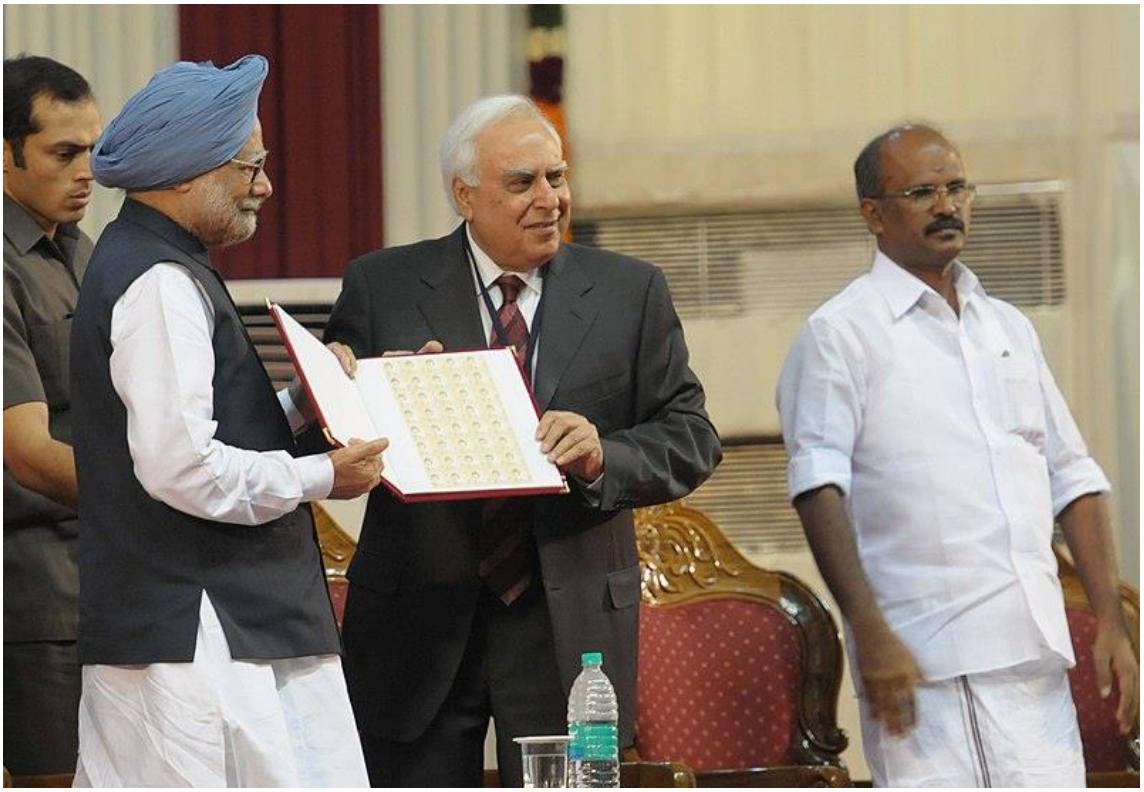
Smt S. Janaki Ammal, wife of Ramanujan


Srinivas Ramanujan
The man who knew infinity and beyond


Ramanujan's Notebooks
Ramanujan's Notebooks
Ramanujan's Notebooks


The Mathematical Genius
SRINIVASA RAMANUJAN
22 December 1887 – 26 April 1920
India's greatest mathematical genius.
Independently compiled nearly 3900 results.
Stated results Ramanujan prime & Ramanujan theta function that inspired a vast amount of further research.
"An equation means nothing to me unless it expresses a thought of God"
Made extraordinary contributions to mathematical analysis, number theory, infinite series, & continued fractions
Known for Ramanujan's sum
Landau–Ramanujan constant
Mock theta functions
Ramanujan conjecture
Ramanujan Soldner constant
Rogers–Ramanujan identities
Ramanujan's master theorem

Ramanujan Museum, Chennai



Manmohan Singh releasing the postal stamp on noted mathematician Shri Srinivas Ramanujan at his 125th Birth Anniversary celebrations.

Group photograph of Profs. G.E. Andrews (holding the Ramanujan Notebooks) with Mrs. Janakimammal, her family members and participants to the International Conference held at Chennai, to mark the birth Centenary of Ramanujan, 22-12-1987".

In the birth Centenary year, 1987, of Ramanujan's, on December 22, in the august presence of the Prime Minister Rajiv Gandhi, these 2 Volumes along with the 'Lost' Notebook of Ramanujan were released and the first copies given to Mrs. Janaki Ammal.

Participants of the Ramanujan Centenary Conference (June 1-5, 1987) at the University of Illinois, Urbana-Champaign.

Prime minister Manmohan Singh presenting the Srinivasa Ramanujan Birth Centenary Award and a gold medal to JNTU-Kakinada Vice-Chancellor.

Prime minister Manmohan Singh presented the prestigious Srinivasa Ramanujan Birth Centenary Award and a gold medal to JNT University-Kakinada (JNTU-K) vice-chancellor Allam Appa Rao at the inaugural session of the 98th Indian Science Congress Association (ISCA) held at SRM University in Chennai today.

The ISCA selected Prof Rao for the award for his contribution to the development of science and technology, specifically in the area of Computational Biology, Software Engineering and Network Security.

Rao has presented 150 research papers at national and international seminars so far and is a member of the international editorial board.

University rector K Satya Prasad, registrar V Ravindra, directors GVR Prasad Raju, B Prabhakar Rao, I Santhi Prabha, K Padma Raju, N Appa Rao, P Vijay Kumar, S Srinivasa Kumar, EV Prasad and Purnanandam and Engineering College Principal P Uday Bhaskar congratulated Rao.

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Links to Websites

Visit these Web Links to know more about Ramanujan

- **Wikipedia article, a good place to start.**

  "Ramanujan independently compiled nearly 3900 results (mostly identities and equations) during his short lifetime. Although a small number of these results were actually false and some were already known, most of his claims have now been proven correct. He stated results that were both original and highly unconventional, such as the Ramanujan prime and the Ramanujan theta function, and these have inspired a vast amount of further research. However, some of his major discoveries have been rather slow to enter the mathematical mainstream. Recently, Ramanujan's formulae have found applications in crystallography and string theory. The Ramanujan Journal, an international publication, was launched to publish work in all areas of mathematics influenced by his work."

- **Review of the book *Ramanujan's lost notebook, Part I* by George E. Andrews and Bruce C. Berndt, published in 2005.**

  "Ramanujan's story is one of the great romantic tales of mathematics. It is an account of triumph and tragedy, of a man of genius who prevailed against incredible adversity and whose life was cut short at the height of his powers. The extent of those powers is only now being fully recognized. Ramanujan had the misfortune to work on problems that, in his time, were considered a mathematical backwater. Modular equations, theta function identities, even continued fractions were viewed as having been played out in the nineteenth century. One might pick

up titbits, but there was nothing important left to be discovered.

"G.H. Hardy knew the error of this view. In his twelve lectures given at Harvard in 1936, he communicated the range and depth of Ramanujan's work."

- **Srinivasa Ramanujan Complete collection of his published papers and unpublished notebooks.**
- **Ramanujan's Notebooks Photographic copy of Ramanujan's first two notebooks.**
- **Relevance of Srinivasa Ramanujan at the dawn of the new millennium by . Srinivasa Rao, chapter from a confercne proceeding, *Number Theory and Discrete Mathematics* (Google books) The chapter begins on page 261. The story about 1729 appears on page 264.**
- **Srinivasa Ramanujan by James R. Newman, chapter in the 4-volume anthology *The World of Mathematics* (Google books) Includes Ramnanujan's 1913 letter to Hardy and some of the formulas in contained.**

"I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'."

- **St. Andrews biography of Ramanujan**

"Ramanujan sailed from India on 17 March 1914. It was a calm voyage except for three days on which Ramanujan was seasick. He arrived in London on 14 April 1914 and was met by Neville. After four days in London they went to Cambridge and

Ramanujan spent a couple of weeks in Neville's home before moving into rooms in Trinity College on 30th April. Right from the beginning, however, he had problems with his diet. The outbreak of World War I made obtaining special items of food harder and it was not long before Ramanujan had health problems.

"Right from the start Ramanujan's collaboration with Hardy led to important results."

- **Srinivasa Ramanujan, biography by Mike Hoffman, a friend of mine at the U. S. Naval Academy.**

  "It is one of the most romantic stories in the history of mathematics: in 1913, the English mathematician G. H. Hardy received a strange letter from an unknown clerk in Madras, India. The ten-page letter contained about 120 statements of theorems on infinite series, improper integrals, continued fractions, and number theory (Here is a .dvi file with a sample of these results). Every prominent mathematician gets letters from cranks, and at first glance Hardy no doubt put this letter in that class. But something about the formulas made him take a second look, and show it to his collaborator J. E. Littlewood. After a few hours, they concluded that the results 'must be true because, if they were not true, no one would have had the imagination to invent them'. "

- **THE RAMANUJAN JOURNAL** The Ramanujan Journal will publish original research papers of the highest quality in all areas of mathematics influenced by Srinivasa Ramanujan. His remarkable discoveries have made a great impact on several

branches of mathematics, revealing deep and fundamental connections.

- **Ramanujan: Essay and Surveys edited by Bruce Berndt and Robert Rankin, 2001 (Google books)**
- **The Ramanujan Pages A collection of very accessible papers about Ramanujan's work by Titus Piezas III (who is this guy?)**
- **Sarah Zubairy, '04, a UR Math graduate who wrote three papers on Ramanujan's work while she was an undergraduate here.**
- **Short video about Ramanujan**
- **Trailer for "The Man Who Knew Infinity."**
- **Ramanujan Pi formula.**

Resource:

https://people.math.rochester.edu/faculty/doug/UGpages/ramanujan.html

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Annotated

Webliography

Webliography Defined

- **Bibliography:**
  Lists books and other printed works referenced in a research paper
- **Webliography:**
  A product born out of web research, represents a list of websites used
- **Annotate:**
  To write a few sentences that both describe and critique the website

Sometimes, we are asked to compile and present our bibliographies to our professors before we turn in our actual essays. We might be asked to comment on our sources to help establish a context for how we will use them, as well as to describe how we have evaluated them as a worthy authority over information. To do this, not only do we supply the list of sources, we write out a paragraph for each of them that details these considerations.

This is called an "annotated bibliography."

If it is done for Web links or websites or webpages, then it is "Annotated Webliography"

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[01] Ramanujan's Notebooks by C Berndt

https://archive.org/details/ramanujans-notebooks/Ramanujan%20Notebooks%20I/

This site contains the details of Ramanujan's Notebooks authored by Bruce C. Berndt Department of Mathematics, University of Illinois Urbana, IL 61801 U.S.A. It contains four volumes: 9 chapters (368 pages) in Vol I; 6 chapters (372 pages) in Vol II; 6 chapters (524 pages) in Vol III; and 10 chapters (231 pages) in Vol IV.

[02] Manuscript Book 2 of Srinivasa Ramanujan

http://www.math.tifr.res.in/~publ/nsrBook2.pdf

This is very rare note book containing XXI chapters, providing his original work in his own handwriting. Go through and enjoy his hand writing as well as mathematical skill. [356 pages]

[03] A Formula of Srinivasa RAMANUJAN

https://core.ac.uk/download/pdf/82155486.pdf

This is a paper by R Sitaramachandra Rao [Department of Mathematics, University of Toledo, Toledo, Ohio 43606, USA] on "A Formula of S Ramanujan". JOURNAL OF NUMBER THEORY 25, 1-19 (1987).

This Paper discusses various equivalent formulations for the sum of an infinite series considered by S. Ramanujan; also, it evaluates, in closed form. various classes of related infinite series.

[04] Srinivasa Ramanujan and Maths in India [PPT]

https://www.aspenphys.org/public/2014_slides/shankar.pdf

Containing 37 PPT slides covering various aspects of S Ramanujan – life, place of living, his notes, his Magic Square, his visit to England, etc.,

[05] Manuscript Book of Srinivasa Ramanujan

https://www.pdfdrive.com/manuscript-book-of-srinivasa-ramanujan-volume-1-d177949115.html

Containing web links to 10 works of S Ramanujan – (1) Manuscript Book of Srinivasa Ramanujan Volume 2; (2) Ramanujan's Lost Notebook: Part I; (3) Ramanujan's Lost Notebook: Part II; (4) Ramanujan Notebooks; (5) Ramanujan's Lost Notebook. Part III; (6) Ramanujan's Lost Notebook: Part V; (7) Ramanujan's Lost Notebook: Part IV; (8) Notebooks of Srinivasa Ramanujan: Volume II; (9) The Man Who Knew Infinity: A Life of the Genius Ramanujan; (10) Collected Papers of Srinivasa Ramanujan.

All ten papers have 4142 pages and have been downloaded 89784 times !

[06] Ramanujan Museum and Education Centre

https://casualwalker.com/museum-for-the-man-who-knew-infinity-ramanujan-museum-royapuram-chennai-travel-guide/

Casual Walker is a curious discovery and recommendations journal that explores a wide range of compelling, thoughtful, and unique local photo stories - connecting travel, traditions, culture, places, temples, art, food & eats, books, events and reviews. We visually walk and guide you to our local world which is full of beauty, adventure, and charm.

To honour this world renounced mathematical genius, Ramanujan Museum was established in 1993 in Chennai by Mr P.K. Srinivasan a Math educator who spent almost 25 years collecting the resources which celebrate Srinivasa Ramanujan's life and massive contributions to the world of mathematicians. The museum was accommodated in the premises of the Avvai Cultural Academy, Royapuram, Chennai by Mr A. T. Bose who is currently head and leading the Ramanujan Museum along with its director Ms Meena Suresh.

To celebrate Ramanujan's birthday on December 22nd, each year the Ramanujan Museum organizes an annual lecture by eminent mathematicians. The Centre also has programmes to spread awareness of mathematics among children.

A visit will be a highly inspirational and informative to learn about the life and works of the great mastermind.

[07] India's greatest Mathematical Genius

https://mathshistory.st-andrews.ac.uk/Biographies/Ramanujan/

MacTutor is a free online resource containing biographies of more than 3000 mathematicians and over 2000 pages of essays and supporting materials. MacTutor is constantly expanding and developing. The list containing the names, alphabetically, chronologically, country wise, birth country wise. Also, this site contains the history of mathematics country wise, in several cultures, topics wise, etc., Very useful to research scholars in mathematics.

[08] Self-taught pure mathematician.

https://www.famousscientists.org/srinivasa-ramanujan/

About 2.3 million years ago our ancestors invented their first primitive tool, the split stone, which they used for cutting and scraping.

Modern humans first appeared about 200,000 years ago. About 50,000 years ago they (or should that be we?) began to use language, symbols, and more complex tools.

As inventions and discoveries added to one another, human civilization, technology, and science advanced and evolved.

This site provides an exhaustive list of international scientists in various fields: physics, mathematics, chemistry, astronomers, biologists, health scientists, scientists in ancient times, etc.,

[09] 200 Greatest Mathematicians

https://fabpedigree.com/james/gmat200.htm

This site provides the Greatest Mathematicians of the Past ranked in approximate order of "greatness." To qualify, the mathematician must be born before 1930 and his work must have breadth, depth, and historical importance. Also, gives the background of ancient mathematicians, early Vedic mathematicians. The site offers the photographs of several mathematicians.

[10] Who was Srinivasa Ramanujan?

https://writings.stephenwolfram.com/2016/04/who-was-ramanujan/

Stephen Wolfram is the creator of Mathematica, Wolfram Alpha and the Wolfram Language; the author of A New Kind of Science; the originator of the Wolfram Physics Project; and the founder and CEO of Wolfram Research. Over the course of more than four decades, he has been a pioneer in the development and application of computational thinking—and has been responsible for many discoveries, inventions and innovations in science, technology and business.

His paper [27 April 2016] was written by him after seeing the film **"The Man Who Knew Infinity"**. The paper, astonishingly, covers the all aspects of Ramanujan's life and his works.

[11] S Ramanujan – Indian Mathematician

https://aptinfo.in/profile-and-life-history-of-srinivasa-ramanujan/

Contains an interesting pictorial profile of Srinivasa Ramanujan. Also, provides a very brief background of Srinivasa Ramanujan. The site provides web links to some other famous mathematicians and scientists.

[12] A Mathematical Legacy

https://www.thoughtco.com/srinivasa-ramanujan-4571004

ThoughtCo is a premier reference site with a 20+ year focus on expert-created education content. We are proud to be one of the top-10 information sites, as measured by comScore, a leading Internet measurement company. In 2018, ThoughtCo received a Communicator Award in the General Education category and a Davey Award in the Education category.

At ThoughtCo, we believe that great inspiration begins with a question, and we help 13 million users answer theirs every month. Whether yours is about science and math, humanities and religion, or architecture and the arts, our in-depth articles, written by literature writers, Ph.Ds., and experienced instructors, are designed to give you the answers and information you need in a clear, easy-to-navigate format. So, whether you are asking for a class, that next conversation, or just because you want to know, ThoughtCo can help.

This web page provides a short biographical details/information of Srinivasa Ramanujan and also, the web links to some famous international mathematicians.

[13] Contribution of India to Mathematics – A Talk [Video]

https://www.youtube.com/watch?v=EcjHccvahHk&ab_channel=IndiaScience

"**The contribution of Indian mathematicians**" was presented in a webinar by renowned mathematician Professor Manjul Bhargava. The webinar was presented by the Department of Science & Technology, government of India.

[14] Indian Mathematics – An Overview [Video]

https://www.youtube.com/watch?v=p2WankcGP3Q&ab_channel=nptelhrd

"Mathematics in India - From Vedic Period to Modern Times" by Prof. M.D.Srinivas, Prof.M.S.Sriram & Prof.K.Ramasubramanian, Department of mathematics, IIT Bombay.

[15] A Brief History of Indian Mathematics

https://www.esamskriti.com/e/Spirituality/Education/A-brief-history-of-Indian-Mathematics-1.aspx

MATHEMATICS has played a significant role in the development of Indian culture for millennia. Mathematical ideas that originated in the Indian subcontinent have had a profound impact on the world. Swami Vivekananda said: 'you know how many sciences had their origin in India. Mathematics began there. You are even today counting 1, 2, 3, etc. to zero, after Sanskrit figures, and you all know that algebra also originated in India.'

A Paper by Mr Vijay Kumar Murthy on "**A brief history of Indian Mathematics**".
The paper elaborately covers the history of Indian Mathematics from 3000 BC to 20th century.

[16] Mathematics in Ancient India – A Paper

https://www.ias.ac.in/article/fulltext/reso/007/04/0004-0019

"**Mathematics in Ancient India**" – a Paper by Amartya Kumar Datta, Associate of Mathematics, Indian Statistical Institute, Kolkata. Paper explores the history of mathematics in India from Vedic period to the present day.

[17] Ancient Mathematical Genius in India

https://www.artofliving.org/in-en/culture/amazing-india/how-ancient-mathemetical-genius-began-in-india

Amazing facts about India.
India is a land where mathematics is coded into everything – from divinity to nature to knowledge to the Vedas and music. Indian philosophy has embraced the concept of zero or *shunya* at one end of the spectrum to *anantha*, infinity, at the other end. The global spread of theses ideas allowed intellectuals around the world to further develop science.

Albert Einstein once said: "We owe a lot to the Indians who taught us how to count, without which no worthwhile scientific discovery could have been made."

What are those revolutionary concepts? Bharath Gyan, a passionate research initiative, is painstakingly dotting the pieces together, foraging amongst lost remnants to keep the glorious past alive and known

[18] List of Famous Indian Mathematicians

https://www.jagranjosh.com/general-knowledge/list-of-famous-indian-mathematicians-from-ancient-to-modern-india-1498193066-1

Mathematics is the study of topics such as quantity (numbers), structure, space, and change. The Aryabhata, Brahmagupta, Mahāvīra, Bhaskara II, Madhava of Sangamagrama, and Nilakantha Somayaji are classical Indian mathematicians whose contributions made them immortal. Here, we are giving the list of Indian mathematicians from Ancient to Modern India their contributions. Provides a list of 53 mathematicians from ancient India to the present day with their period of existence and their contribution to mathematics.

[19] Top TEN Most Intelligent People on Earth

https://www.jagranjosh.com/general-knowledge/top-10-most-intelligent-people-on-earth-1477392275-1

Without geniuses History can't be created. Persons like Newton, Thomas Alva Edison, Leonardo da Vinci, Galileo Galilei etc are still alive due to their methods, ideas and formulas. There are also people who have gained fame and achieved in small age with their mind-boggling IQ levels, exceptionally intelligent and talented including brilliantly accomplished academics, former child prodigies etc.

We can also call them geeks, nerds or superb sharp intellectuals. Here are 10 most intelligent people on earth those who have created history.

[20] S RAMANUJAN: The Mathematician & His Legacy [Video]

https://www.google.com/search?rlz=1C1CHZN_enIN990IN991&sxsrf=AJOqlzWUgYPAX9K7AFFiGmoazSCRKHzu8w:1676714424393&source=lnms&tbm=vid&sa=X&ved=2ahUKEwjMiZ_95579AhV303MBHfofDNUQ_AUoBHoECAEQBg&q=srinivasa%20ramanujan%20videos&biw=1350&bih=640&dpr=1#fpstate=ive&vld=cid:40fe85dd,vid:uhNGCn_3hmc

Each year, December 22nd is celebrated as the National Mathematical Day in India to commemorate the birth anniversary of the self-taught genius – Srinivasa Ramanujan whose work continues to intrigue, inspire and challenge mathematicians around the world. His fertile mind envisioned a world where science and mathematics converge to uncover fundamental principles governing our universe. Nearly a hundred years after his death, mathematics has slowly developed tools that are unravelling some of the secrets behind his intuitive suppositions. His oeuvre has opened up new vistas of possibilities, new branches of mathematics have sprouted and proliferated and found application in different fields of sciences, implausible in his own times. **The Film brings together his life on screen, with period enactments, interviews and animation.**

[21] Mrs Janakiammal (Wife of S Ramanujan) [Video]

https://www.youtube.com/watch?v=1qaRtMjTD-k&ab_channel=Kasturi

This is a rare and wonderful video covering an interview with Mrs Janakiammal, wife of great mathematician S Ramanujan. She vividly narrates her life and experiences with her husband.

[22] Ramanujan: The Man Who Knew Infinity

https://www.indiascienceandtechnology.gov.in/listingpage/ramanujan-man-who-knew infinity#:~:text=Ramanujan's%20contribution%20extends%20to%20mathematical,of%20many%20algorithms%20used%20today.

This information from the Department of Science and Technology, government of India, providing all important information and other details about S Ramanujan. Also, it contains several interesting information about him and other related areas of interest about mathematics and other areas connected to S Ramanujan.

[23] Ramanujan Prize

https://www.indiascienceandtechnology.gov.in/india-international-fellows?combine=Ramanujan%20Prize

This information is from the Department of Science and Technology, government of India, providing the details about Ramanujan Prize and the list of recipients.
The government of India has taken several initiatives towards the international exposure of scientists working in India in addition to providing lucrative incentives to bring back the eminent scientists to the country vis-a-vis implementation of several International Fellowship Programmes / Schemes. The list of Ramanujan Prize winners includes the names of the awardees, their affiliation, year of the award and area of specialisation.

[24] Ramanujan Fellowship

https://www.indiascienceandtechnology.gov.in/programme-schemes/research-and-development/ramanujan-fellowship

This information is from the Department of Science and Technology, government of India, providing the details about Ramanujan Fellowship and the list of recipients.

These International Fellowships include:

i) Fellowships for attracting highly skilled, outstanding researchers and scientists of Indian origin working abroad by providing them an attractive avenue to pursue their R&D interests in Indian Institutions.

ii) Fellowships providing opportunities to the best and brightest Indian students/ researchers / scientists to go abroad and gain exposure to world class research facilities and peers in leading institutions, thereby helping to build long-term R&D linkages and collaborations.

iii) Besides these, Indian scientists regularly apply for other international fellowships and their contributions are significant in the international arena.

These efforts have helped develop a pool of talents who are helping in the furtherance of excellence in science in the country. This Section provides information regarding the scholars honoured with these International Fellowships, the year of award, their affiliations and their area of work. It includes information regarding scholars like Ramanujan Fellows, Ramalingaswami

Fellows, Wellcome Trust/DBT India Alliance Fellows, Fulbrighters, Human Frontier Science Program (HFSP) Fellows, Indo-Australia Early and Mid-Career Researchers (EMCR) and so on...

The scheme provides support to active researchers/ scientists/engineers who want to return to India from abroad and contribute their work for the country.

Programme Type: Research and Development
Ministry / Department: Department of Science & Technology (DST), Govt of India
Focus Area: All Areas of Science
Target Audience: Scientists and Engineers
Funding Agency: Science and Engineering Research Board (SERB)
Eligibility: Indian scientists and engineers working abroad and are below the age of 40 years. The Nominee should possess higher degree or equivalent, such as Ph.D. in Science/ Engineering, Masters in Engineering or Technology/ MD in Medicine, etc. and have adequate professional experience.
Purpose: Capacity Building, Research
Funding Cycle: Rolling
Duration: 5 Years
Application Method: Online
Fellowship / Financial Assistance: Fellowship will be Rs. 1,35,000/- per month (consolidated including HRA). A research grant of Rs.7.00 lakh per annum and Rs.60,000/- per annum as overhead charges

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RAMANUJAN, HIS LOST NOTEBOOK, ITS IMPORTANCE

BRUCE C. BERNDT

1. Brief Biography of Ramanujan

Throughout history most famous mathematicians were educated at renowned centers of learning and were taught by inspiring teachers, if not by distinguished research mathematicians. The one exception to this rule is Srinivasa Ramanujan, born on December 22, 1887 in Erode in the southern Indian State of Tamil Nadu. He lived most of his life in Kumbakonam, located to the east of Erode and about 250 kilometers south, southwest of Madras, or Chennai as it is presently named. At an early age, he won prizes for his mathematical ability, not mathematics books as one might surmise, but books of English poetry, reflecting British colonial rule at that time. At the age of about fifteen, he borrowed a copy of G. S. Carr's *Synopsis of Pure and Applied Mathematics*, which was the most influential book in Ramanujan's development. Carr was a tutor and compiled this compendium of approximately 4000–5000 results (with very few proofs) to facilitate his tutoring at Cambridge and London. One or two years later, Ramanujan entered the Government College of Kumbakonam, often called "the Cambridge of South India," because of its high academic standards. By this time, Ramanujan was consumed by mathematics and would not seriously study any other subject. Consequently he failed his examinations at the end of the first year and lost his scholarship. Because his family was poor, Ramanujan was forced to terminate his formal education.

At about the time Ramanujan entered college, he began to record his mathematical discoveries in notebooks. Living in poverty with no means of financial support, suffering at times from serious illnesses (including two long bouts of dysentery), and working in isolation, Ramanujan devoted all of his efforts in the next five years to mathematics, while continuing to record his theorems without proofs in notebooks.

In 1909, Ramanujan married S. Janaki, who was only nine years old. More pressure was therefore put upon him to find a job, and so in 1910 he arranged a meeting with V. Ramaswami Aiyar, who had founded the *Indian Mathematical Society* three years earlier and who was working as a Deputy Collector in the Civil Service. After Ramanujan showed Ramaswami Aiyar his notebooks, the latter contacted R. Ramachandra Rao, Collector in the town of Nellore, north of Madras, who agreed to provide Ramanujan with a monthly stipend so that he could continue to work unceasingly on mathematics and not worry about employment.

In 1910, with the financial support of Ramachandra Rao, Ramanujan moved to Madras. For reasons that are unclear, after fifteen months, Ramanujan declined further support and subsequently became a clerk in the Madras Port Trust Office, where

he was encouraged, especially by Sir Francis Spring and S. Narayana Aiyar, Chairman and Chief Accountant, respectively. Narayana Aiyar held a Master's Degree in Mathematics and at that time was probably one of the most capable mathematicians in India [20]. They persuaded Ramanujan to write English mathematicians about his mathematical discoveries. One of them, G. H. Hardy, responded encouragingly and invited Ramanujan to come to Cambridge to develop his mathematical gifts. Ramanujan's family were Iyengars, a conservative orthodox branch in the Brahmin tradition, and especially his mother was adamantly opposed to her son's "crossing the seas" and thereby becoming "unclean." After overcoming family reluctance, Ramanujan boarded a passenger ship for England on March 17, 1914.

At about this time, Ramanujan evidently stopped recording his theorems in notebooks. That Ramanujan no longer concentrated on logging entries in his notebooks is evident from two letters that he wrote to friends in Madras during his first year in England [31, pp. 112–113; 123–125]. In a letter of November 13, 1914 to his friend R. Krishna Rao, Ramanujan confided, "I have changed my plan of publishing my results. I am not going to publish any of the old results in my notebooks till the war is over." And in a letter of January 7, 1915 to S. M. Subramanian, Ramanujan admitted, "I am doing my work very slowly. My notebook is sleeping in a corner for these four or five months. I am publishing only my present researches as I have not yet proved the results in my notebooks rigorously."

On March 24, 1915, near the end of his first winter in Cambridge, Ramanujan wrote to his friend E. Vinayaka Row [31, pp. 116–117] in Madras, "I was not well till the beginning of this term owing to the weather and consequently I couldn't publish any thing for about 5 months." By the end of his third year in England, Ramanujan was critically ill, and, for the next two years, he was confined to nursing homes. After World War I ended, Ramanujan returned home in March, 1919, but his health continued to deteriorate, and on April 26, 1920 Ramanujan died at the age of 32.

In both England and India, Ramanujan was treated for tuberculosis, but his symptoms did not match those of the disease. More recently, an English physician, D. A. B. Young [32, pp. 65–75] carefully examined all extant records and symptoms of Ramanujan's illness and convincingly concluded that Ramanujan suffered from hepatic amoebiasis, a parasitic infection of the liver. Amoebiasis is a protozoal infection of the large intestine that gives rise to dysentery. Relapses occur when the host–parasite relationship is disturbed, which likely happened when Ramanujan entered a colder climate. The illness is very difficult to diagnose, but once diagnosed, it can be cured.

Despite being confined to nursing homes for two of his five years in England, Ramanujan made enormously important contributions to mathematics, several in collaboration with Hardy, which, although they won him immediate and lasting fame, are probably recognized and appreciated more so today than they were at that time. Most of Ramanujan's discoveries lie in the areas of (primarily) number theory, analysis, and combinatorics, but they influence many modern branches of both mathematics and physics.

2. History of Ramanujan's Notebooks

After Ramanujan died, Hardy strongly urged that Ramanujan's notebooks be edited and published. Hardy used the word "edit" in the broader sense to indicate that each claim made by Ramanujan in his notebooks should be examined and proved, if a proof did not already exist. Of the three notebooks that Ramanujan left us, the second is the most extensive and is actually a revision of the first, although there are over 100 entries in the first notebook that cannot be found in the second. The third notebook has only 33 pages. Altogether, the notebooks contain about 3200–3300 different entries, almost all of them without proofs and most of them original discoveries. We do not know for certain when Ramanujan began to record his results in notebooks, but as mentioned above, we think his compilation began in about 1904 when he entered college; as the letters above indicate, he ceased to record his theorems in notebooks when he departed for England in 1914.

At the University of Madras, various papers and handwritten copies of all three notebooks were sent to Hardy in 1923 with the intent of bringing together all of Ramanujan's work for publication. It transpired that Ramanujan's *Collected Papers* [61] were published in 1927, but his notebooks and other manuscripts were not published.

Sometime in the late 1920s, two English mathematicians, G. N. Watson and B. M. Wilson, undertook the task of editing Ramanujan's notebooks. Wilson died prematurely in 1935, and although Watson worked for 10–15 years on the task and wrote over 30 papers inspired by Ramanujan's mathematics, the work was never completed.

It was not until 1957 that the notebooks were made available to the public when the Tata Institute of Fundamental Research in Bombay published a photocopy edition [62], but no editing was undertaken.

In February, 1974 while reading two papers by Emil Grosswald [47], [48], in which some formulas from the notebooks were proved, the present author observed that he could prove these formulas by using a theorem he had proved two years earlier and so was naturally curious to determine if there were other formulas in the notebooks that could be proved using his theorem. He was at the Institute for Advanced Study in Princeton at that time, and fortunately, the library at Princeton University had a copy of the Tata Institute's edition. He found a few more formulas, similar to those that he had already proven, but he found a few thousand other formulas that he could not prove. In the following two years the author wrote two long papers [12], [13] on series identities and evaluations in the spirit of those formulas from the notebooks that Grosswald had first proved.

All of the aforementioned entries can be found in Chapter 14 of Ramanujan's second notebook. After the spring semester at the University of Illinois ended in May, 1977, the author decided to attempt to find proofs for all 87 formulas in Chapter 14. He worked on this project for over a year, and then G. E. Andrews from Pennsylvania State University visited the University of Illinois and informed him that on a visit to Trinity College Library at Cambridge two years earlier, he learned that Watson and Wilson's efforts in editing the notebooks were preserved there. The librarian at Trinity College kindly sent the author a copy of these notes, and so with their help on certain chapters, he began to devote all of his research time toward proving the theorems stated

by Ramanujan in his three notebooks. With the assistance of several mathematicians, the task was completed in five volumes [14]–[18] over a period of 21 years.

Why did not Ramanujan record any of his proofs in his three notebooks? There are perhaps several reasons. First, Ramanujan was perhaps influenced by the style of Carr's book in which one theorem after another is stated without proofs. Second, like most Indian students in his time, Ramanujan worked primarily on a slate. Paper was expensive, and so writing out all of his proofs on paper would have been prohibitive for Ramanujan. Therefore, after rubbing out the proofs on his slate with his sleeve, Ramanujan recorded only the final results in his notebooks. Third, Ramanujan probably felt that there was no need to record his arguments. If someone asked him how to prove a particular result in his notebooks, he undoubtedly could supply his proof. He had never intended that his notebooks would be made available to the mathematical public. They were his own personal compilation of what he had discovered. It has also been proposed that Ramanujan did not provide proofs because he was following the long-standing Indian tradition of stating mathematical facts without arguments or proofs. However, there is no evidence that Ramanujan had studied ancient Indian mathematics. With India being under British colonial rule at that time, Ramanujan was more influenced by English mathematics than by traditional Indian mathematics. In fact, Ramanujan's early papers and problems that he contributed to the *Journal of the Indian Mathematical Society* were in the same milieu as the mathematics being studied and created by other Indian mathematicians in the period around 1910. We emphasize, however, that Ramanujan's work did not remain at this level, as he quickly took the modest amount of mathematics that he knew and developed it in many directions, so that mathematicians today continue to build along the paths that he started.

Speculations about Ramanujan's methods and proofs are plentiful. Many have suggested that he discovered his results by "intuition," or by making deductions from numerical calculations, or by inspiration from Goddess Namagiri. Indeed, like many mathematicians, Ramanujan evidently made extensive calculations. However, explanations emphasizing "intuition," religion, or Eastern mysticism are of no value. Ramanujan's idea of a proof arose from the British tradition and not from the venerable Indian tradition. Because of his lack of a rigorous training in mathematics, as Ramanujan himself was aware, some of his arguments were not rigorous by then contemporary standards. Nonetheless, despite his lack of rigor at times, Ramanujan doubtless thought and devised proofs like any other mathematician, but with insights that surpass all but a few of the greatest mathematicians.

3. Discovery of the Lost Notebook

As mentioned above, in the spring of 1976, Andrews visited Trinity College Library at Cambridge University. Dr. Lucy Slater had suggested to him that there were materials deposited there from the estate of the late G. N. Watson that might be of interest to him. In one box of materials from Watson's estate, Andrews found several items written by Ramanujan, with the most interesting item being a manuscript written on 138 sides in Ramanujan's distinctive handwriting. The sheets contained over six

hundred formulas without proofs. Although technically not a notebook, and although technically not "lost," as we shall see in the sequel, it was natural in view of the fame of Ramanujan's (earlier) notebooks [62] to call this manuscript *Ramanujan's lost notebook*. Almost certainly, this manuscript, or at least most of it, was written during the last year of Ramanujan's life, after his return to India from England.

The manuscript contains no introduction or covering letter. In fact, there are hardly any words in the manuscript. There are a few marks evidently made by a cataloguer, and there are also a few remarks in the handwriting of Hardy. Undoubtedly, the most famous objects examined in the lost notebook are the *mock theta functions*, about which more will be written later.

The natural, burning question now is: *How did this manuscript of Ramanujan come into Watson's possession?* We think that the manuscript's history can be traced.

4. History of the Lost Notebook

After Ramanujan died on April 26, 1920, his notebooks and unpublished papers were given by his widow, Janaki, to the University of Madras. (It should be remarked that in a conversation with the author, Janaki told him that during the funeral of her late husband, many of his papers were stolen by two certain people. These stolen papers have never been located.) Also at that time, Hardy strongly advocated bringing together all of Ramanujan's manuscripts, both published and unpublished, for publication. On August 30, 1923, Francis Dewsbury, the registrar at the University of Madras, wrote to Hardy informing him that [31, p. 266]:

> I have the honour to advise despatch to-day to your address per registered and insured parcel post of the four manuscript note-books referred to in my letter No. 6796 of the 2nd idem.
>
> I also forward a packet of miscellaneous papers which have not been copied. It is left to you to decide whether any or all of them should find a place in the proposed memorial volume. Kindly preserve them for ultimate return to this office.

(Hardy evidently never returned any of the miscellaneous papers.) Although no accurate record of this material exists, the amount sent to Hardy was doubtless substantial. It is therefore highly likely that this "packet of miscellaneous papers" contained the aforementioned "lost notebook." R. A. Rankin, in fact, opines [32, p. 124]:

> It is clear that the long MS represents work of Ramanujan subsequent to January 1920 and there can therefore be little doubt that it constitutes the whole or part of the miscellaneous papers dispatched to Hardy from Madras on 30 August 1923.

Further details can be found in Rankin's accounts of Ramanujan's unpublished manuscripts [65], [66], [32, pp. 117–142].

In 1934, Hardy passed on to Watson a considerable amount of his material on Ramanujan. However, it appears that either Watson did not possess the "lost" notebook in 1936 and 1937 when he published his papers [70], [71] on mock theta functions, or he had not thoroughly examined it. In any event, Watson [70, p. 61], [31, p. 330] writes

that he believes that Ramanujan was unaware of certain third order mock theta functions and transformation formulas for the fifth order mock theta functions. But, in his lost notebook, Ramanujan did indeed examine these third order mock theta functions, and found transformation formulas for the fifth order mock theta functions. Watson's interest in Ramanujan's mathematics waned in the late 1930s, and Hardy died in 1947. In conclusion, sometime between 1934 and 1947 and probably closer to 1947, Hardy gave Watson the manuscript that we now call the "lost notebook."

Watson was Mason Professor of Pure Mathematics at the University of Birmingham for most of his career, retiring in 1951. He died in 1965 at the age of 79. Rankin, who succeeded Watson as Mason Professor but who had since become Professor of Mathematics at the University of Glasgow, was asked to write an obituary of Watson for the London Mathematical Society. Rankin wrote [65], [32, p. 120]:

> For this purpose I visited Mrs Watson on 12 July 1965 and was shown into a fair-sized room devoid of furniture and almost knee-deep in manuscripts covering the floor area. In the space of one day I had time only to make a somewhat cursory examination, but discovered a number of interesting items. Apart from Watson's projected and incomplete revision of Whittaker and Watson's *Modern Analysis* in five or more volumes, and his monograph on *Three decades of midland railway locomotives*, there was a great deal of material relating to Ramanujan, including copies of Notebooks 1 and 2, his work with B. M. Wilson on the Notebooks and much other material. ... In November 19 1965 Dr J. M. Whittaker who had been asked by the Royal Society to prepare an obituary notice [72], paid a similar visit and unearthed a second batch of Ramanujan material. A further batch was given to me in April 1969 by Mrs Watson and her son George.

Since her late husband had been a Fellow and Scholar at Trinity College and had had an abiding, lifelong affection for Trinity College, Mrs. Watson agreed with Rankin's suggestion that the library at Trinity College would be the most appropriate place to preserve her husband's papers. Since Ramanujan had also been a Fellow at Trinity College, Rankin's suggestion was even more appropriate.

During the next three years, Rankin sorted through Watson's papers, and dispatched Watson's and Ramanujan's papers to Trinity College in three batches on November 2, 1965; December 26, 1968; and December 30, 1969, with the Ramanujan papers being in the second shipment. Rankin did not realize the importance of Ramanujan's papers, and so when he wrote Watson's obituary [64] for the *Journal of the London Mathematical Society*, he did not mention any of Ramanujan's manuscripts. Thus, for almost eight years, Ramanujan's "lost notebook" and some fragments of papers by Ramanujan lay in the library at Trinity College, known only to a few of the library's cataloguers, Rankin, Mrs. Watson, Whittaker, and perhaps a few others. The 138-page manuscript waited there until Andrews found it and brought it before the mathematical public in the spring of 1976. It was not until the centenary of Ramanujan's birth on December 22, 1987, that Narosa Publishing House in New Delhi published in photocopy form Ramanujan's lost notebook and his other unpublished papers [63].

5. The Origin of the Lost Notebook

Having detailed the probable history of Ramanujan's lost notebook, we return now to our earlier claim that the lost notebook was written in the last year of Ramanujan's life. On February 17, 1919, Ramanujan returned to India after almost five years in England, the last two being confined to nursing homes. Despite the weakening effects of his debilitating illness, Ramanujan continued to work on mathematics. Of this intense mathematical activity, up to the discovery of the lost notebook, the mathematical community knew only of the mock theta functions. These functions were described in Ramanujan's last letter to Hardy, dated January 12, 1920 [61, pp. xxix–xxx, 354–355], [31, pp. 220–223], where he wrote:

> I am extremely sorry for not writing you a single letter up to now .... I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

In this letter, Ramanujan defines four third order mock theta functions, ten fifth order functions, and three seventh order functions. He also includes three identities satisfied by the third order functions and five identities satisfied by his first five fifth order functions. He states that the other five fifth order functions also satisfy similar identities. In addition to the definitions and formulas stated by Ramanujan in his last letter to Hardy, the lost notebook contains further discoveries of Ramanujan about mock theta functions. In particular, it contains the five identities for the second family of fifth order functions that were only mentioned but not stated in the letter.

We think that we have made the case for our assertion that the lost notebook was composed during the last year of Ramanujan's life, when, by his own words, he discovered the mock theta functions. In fact, only a fraction (perhaps 5–8%) of the notebook is devoted to the mock theta functions themselves.

6. General Content of the Lost Notebook

The next fundamental question is: *What is in Ramanujan's lost notebook besides mock theta functions?* A majority of the results fall under the purview of q-series. These include mock theta functions, theta functions, partial theta functions, false theta functions, identities connected with the Rogers–Fine identity, several results in the theory of partitions, Eisenstein series, modular equations, the Rogers–Ramanujan continued fraction, other q-continued fractions, asymptotic expansions of q-series and q-continued fractions, integrals of theta functions, integrals of q-products, and incomplete elliptic integrals. Other continued fractions, other integrals, infinite series identities, Dirichlet series, approximations, arithmetic functions, numerical calculations, Diophantine equations, and elementary mathematics are some of the further topics examined by Ramanujan in his lost notebook.

The Narosa edition [63] contains further unpublished manuscripts, portions of both published and unpublished papers, letters to Hardy written from nursing homes, and

scattered sheets and fragments. The three most famous of these unpublished manuscripts are those on the partition function and Ramanujan's tau function [30], [6], forty identities for the Rogers–Ramanujan functions [26], [6], and the unpublished remainder of Ramanujan's published paper on highly composite numbers [58], [61, pp. 78–128], [6].

In the passages that follow, we select certain topics and examples to illustrate the content and importance of Ramanujan's discoveries found in his lost notebook. Andrews and the author are in the process of writing five volumes on the lost notebook that are analogous to those that the author wrote on the earlier notebooks [14]–[18]. At the moment of this writing, four of the volumes [4]–[7] have been published.

7. q-Series and Theta Functions

The vast majority of entries in Ramanujan's lost notebook are on q-series. Thus, we should begin the more technical portion of this essay by giving a brief introduction to q-series. Generally, a q-series has expressions of the type

$$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \geq 0, \tag{7.1}$$

in the summands, where we interpret $(a;q)_0 = 1$. If the *base* q is understood, we often use the notation at the far left-hand side of (7.1). Series with factors $(a;q)_n$ in their summands are also called *Eulerian series*. Our definition of a q-series is not entirely satisfactory, because often in the theory of q-series, we let parameters in the summands tend to 0 or to ∞, and consequently it may happen that no factors of the type $(a;q)_n$ remain in the summands. In some cases, what may remain is a theta function. Following the lead of Ramanujan, we define a general theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{7.2}$$

Perhaps the most useful property of theta functions is the famous Jacobi triple product identity [16, p. 35, Entry 19] given by

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \qquad |ab| < 1, \tag{7.3}$$

where

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n, \qquad |q| < 1.$$

For this exposition, in Ramanujan's notation and with the use of (7.3), only one special case,

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{7.4}$$

is relevant for us. If $q = e^{2\pi i\tau}$, where $\operatorname{Im}\tau > 0$, then $q^{1/24} f(-q) = \eta(\tau)$, the Dedekind eta function. The last equality in (7.4) renders Euler's pentagonal number theorem. For a more detailed introduction to q-series, see the author's paper [19].

8. Mock Theta Functions

As indicated above, mock theta functions are certain kinds of q-series first introduced by Ramanujan in his last letter to Hardy written on 12 January 1920. He begins his letter by examining the asymptotic behavior of two q-series as $t \to 0$, where $q = e^{-t}$. For example,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \sqrt{\frac{t}{2\pi}} \exp\left(\frac{\pi^2}{6t} - \frac{t}{24}\right) + o(1), \tag{8.1}$$

as $t \to 0$. The series above is the reciprocal of a theta function, namely, it is the reciprocal of $f(-q)$, defined in (7.4). Ramanujan then asks if the converse is true. That is, suppose we have a q-series that exhibits an asymptotic behavior of the kind described in (8.1) as we approach any exponential singularity $e^{2\pi i m/n}$ of the function. Must the function actually be a theta function plus some easily described trivial function? Ramanujan says "not necessarily so." "When it is not so I call the function Mock ϑ-function."

Ramanujan then gives several examples of mock theta functions. For example, he defines

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \tag{8.2}$$

and asserts that

$$f(q) + \sqrt{\frac{\pi}{t}} \exp\left(\frac{\pi^2}{24t} - \frac{t}{24}\right) \to 4, \tag{8.3}$$

as $t \to 0$, with $q = e^{-t}$. Then he remarks, "It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$. (The definition of $f(q)$ in (8.2) has no relation with the function $f(-q)$ defined in (7.4).) Thus, $f(q)$ is a mock theta function (of the third order). What is the *order* of a mock theta function? Ramanujan does not tell us. We emphasize that Ramanujan does not *prove* that $f(q)$ is actually a mock theta function according to his somewhat imprecise definition. Moreover, *no one* since has actually *proved* this statement, nor has anyone proved that *any* of Ramanujan's mock theta functions are really mock theta functions according to his definition. Note that the series on the left side of (8.1) is similar in appearance to the series defining $f(q)$ in (8.2); only the signs of the parameters in the summands' q-products are different. However, the two series behave quite differently, both analytically and arithmetically in regard to their coefficients. Indeed, one of the fascinating features of q-series is that making what appears to be a small modification in the series terms drastically alters the behavior of the function. Only recently, A. Folsom, K. Ono, and R. C. Rhoades [45] established for the first time that the limit on the right-hand side of (8.3) is indeed equal to 4.

Ramanujan's lost notebook contains many identities involving mock theta functions. We offer two identities for fifth order mock theta functions:

$$\chi(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1})_{n+1}} \tag{8.4}$$

and

$$f_0(q) + 2\Phi(q^2) = \frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{8.5}$$

where

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \qquad \Phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}.$$

All three functions $\chi(q)$, $f_0(q)$, and $\Phi(q)$ are mock theta functions. Both of these identities have interesting implications in the theory of partitions, which we address in Section 9.

The discovery of the lost notebook by Andrews and then the publication of the lost notebook by Narosa in 1988 [63] stimulated an enormous amount of research on mock theta functions, as researchers found proofs of the many mock theta function identities found in the lost notebook. We mention only a few of the more important contributions by: Andrews [2], [3], Andrews and F. Garvan [9], and D. Hickerson [52] on fifth order mock theta functions; Andrews and Hickerson [10] on sixth order mock theta functions; Andrews [2] and Hickerson [53] on seventh order mock theta functions; and Y.–S. Choi [37]–[40] on tenth order mock theta functions.

As we have seen, Ramanujan's definition of a mock theta function is somewhat vague. Can a precise, coherent theory be developed and find its place among the other great theories of our day? In 1987, at a meeting held at the University of Illinois commemorating Ramanujan on the centenary of his birth, F. J. Dyson addressed this question [42, p. 20], [32, p. 269].

> The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions. Perhaps we may one day see a preprint written by a physicist with the title "Mock Atkin-Lehner Symmetry." But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal farther.

Since we invoke the words, *modular form*, at times in the sequel, we provide here a brief definition. Let $V(\tau) = (a\tau + b)/(c\tau + d)$, where a, b, c, and d are integers such that $ad - bc = 1$, and where $\operatorname{Im} \tau > 0$. Then $f(\tau)$ is a modular form of weight k if

$$f(V(\tau)) = \epsilon(a,b,c,d)(c\tau + d)^k f(\tau), \tag{8.6}$$

where k is real (usually an integer or half of an integer) and $|\epsilon(a,b,c,d)| = 1$.

In recent years, the work of S. Zwegers [73] and K. Bringmann and K. Ono [35], [36] has made progress in the direction envisioned by Dyson. First it was observed, to take one example, that the infinite product on the right-hand side of (8.5) essentially

coincided with the Fourier expansion of a certain weakly holomorphic modular form, where the term "weakly holomorphic" indicates that the modular form is analytic in the upper half-plane, but may have poles at what are called "cusps" on the real axis. In his doctoral dissertation [73], Zwegers related mock theta functions to real analytic vector-valued modular forms by adding to Ramanujan's mock theta functions certain non-holomorphic functions, which are called period integrals. Earlier work of Andrews [2], who used Bailey pairs to express Ramanujan's Eulerian series in terms of Hecke-type series, was also essential for Zwegers, since he applied his ideas to the Hecke-type series rather than Ramanujan's original series. Zwegers' real analytic modular forms are examples of *harmonic Maass forms*. Briefly, a Maass form satisfies the functional equation (8.6) with $k = 0$ and is an eigenfunction of the hyperbolic Laplacian

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \tag{8.7}$$

where $\tau = x + iy$. A harmonic Maass form $M(\tau)$ again satisfies a functional equation of the type (8.6) when k is an integer and a slightly different functional equation if k is half of an integer, but the operator (8.7) is replaced by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right).$$

It is now required that $\Delta_k M = 0$.

Returning to (8.2), Bringmann and Ono [36] examined the more general function

$$R(\omega, q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q; q)_n (\omega^{-1} q; q)_n} \tag{8.8}$$

and proved (under certain hypotheses), when ω is a root of unity, that $R(\omega, q)$ is the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass form. In general, Bringmann and Ono showed that each of Ramanujan's mock theta functions is the holomorphic part of a harmonic Maass form.

Recall that in their famous paper [50], Hardy and Ramanujan developed an asymptotic series for the partition function $p(n)$, defined to be the number of ways the positive integer n can be expressed as a sum of positive integers. For example, since $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, $p(4) = 5$. Some years later, improving on their work, H. Rademacher [57] found a convergent series representation for $p(n)$. If we write the mock theta function $f(q)$ from (8.2) as $f(q) = \sum_{n=0}^{\infty} \alpha(n) q^n$, Andrews [1] analogously found an asymptotic series for $\alpha(n)$. Bringmann and Ono [35] were able to replace the asymptotic formula by an exact formula confirming a conjecture of Andrews.

We have sketched only a few highlights among the extensive recent developments in the theory of mock theta functions. Readers are encouraged to read Ono's comprehensive description [56] of these developments. In Ono's paper, readers will also find discussions and references to the permeation of mock theta functions in physics, thus providing evidence for Dyson's prophetic vision.

9. Partitions

Recall from above the definition of the partition function $p(n)$. Inspecting a table of $p(n)$, $1 \le n \le 200$, calculated by P. A. MacMahon, Ramanujan was led to conjecture the congruences

$$\begin{cases} p(5n+4) & \equiv 0 \pmod 5, \\ p(7n+5) & \equiv 0 \pmod 7, \\ p(11n+6) & \equiv 0 \pmod{11}, \end{cases} \tag{9.1}$$

which he eventually proved [60], [63], [30]. In 1944, Dyson [41] sought to combinatorially explain (9.1) and in doing so defined the *rank of a partition* to be *the largest part minus the number of parts*. For example, the rank of 3+1 is 1. Dyson observed that the congruence classes for the rank modulo 5 and 7 divided the partitions of $p(5n+4)$ and $p(7n+5)$, respectively, into equinumerous classes. These conjectures were subsequently proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [11]. However, for the third congruence in (9.1), the corresponding criterion failed, and so Dyson conjectured the existence of a statistic, which he called the *crank* to combinatorially explain the congruence $p(11n+6) \equiv 0 \pmod{11}$. The *crank of a partition* was found by Andrews and Garvan [8] and is defined to be *the largest part if the partition contains no one's, and otherwise to be the number of parts larger than the number of one's minus the number of one's*. The crank divides the partitions into equinumerous congruence classes modulo 5, 7, and 11 for the three congruences, respectively, in (9.1).

At roughly the same time that Andrews and Garvan found the crank, it was observed that in his lost notebook, Ramanujan had found the generating functions for both the rank and the crank. First, if $N(m,n)$ denotes the number of partitions of n with rank m, then

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n}. \tag{9.2}$$

The definition (9.2) should be compared with that in (8.8). Second, if $M(m,n)$ denotes the number of partitions of n with crank m, then, except for a few small values of m and n,

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n) z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}. \tag{9.3}$$

We do not know if Ramanujan knew the combinatorial implications of the rank and crank. However, in view of the several results on the generating functions for the rank and crank as well as calculations for cranks found in his lost notebook, it is clear that he had realized the importance of these two functions [46], [23]. There is also evidence that his very last mathematical thoughts were on cranks before he died on April 26, 1920 [24].

Many identities in the lost notebook have partition theoretic implications. As promised earlier, we now examine the partition-theoretic interpretations of (8.4) and (8.5).

We state (8.4) in an equivalent form: The number of partitions of a positive integer N where the smallest part does not repeat and the largest part is at most twice the

smallest part equals the number of partitions of N where the largest part is odd and the smallest part is larger than half the largest part. As an example, take $N = 7$. Then the relevant partitions are, respectively, $7 = 4 + 3 = 2 + 2 + 2 + 1$ and $7 = 3 + 2 + 2 = 1 + 1 + 1 + 1 + 1 + 1 + 1$. A short proof can be constructed with the use of Ferrers diagrams, and we leave this proof for the reader.

To examine (8.5), we first define $\rho_0(n)$ to be the number of partitions of n with unique smallest part and all other parts $\leq$ the double of the smallest part. For example, $\rho_0(5) = 3$, with the relevant partitions being 5, 3+2, and 2+2+1. Second, let $N(a, b, n)$ denote the number of partitions of n with rank congruent to a modulo b. Then (8.5) is equivalent to *The First Mock Theta Conjecture*,

$$N(1, 5, 5n) = N(0, 5, 5n) + \rho_0(n). \tag{9.4}$$

For example, if $n = 5$, then $N(1, 5, 25) = 393$, $N(0, 5, 25) = 390$, and, as observed above, $\rho_0(5) = 3$. Although (8.5) has been proved by Hickerson, and now also by A. Folsom [43] and by Hickerson and E. Mortenson [54], a combinatorial proof of (9.4) has never been given.

10. Further q-Series

We have discussed only two facets among Ramanujan's voluminous contributions to q-series in his lost notebook. In this short section, we briefly provide two more examples in illustration of this abundance. One of the showpieces in the theory of q-series is Heine's transformation [5, p. 6, Thm. 1.2.1]

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} t^n = \frac{(b; q)_\infty (at; q)_\infty}{(c; q)_\infty (t; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n (t)_n}{(at)_n (q)_n} b^n, \tag{10.1}$$

where $|t|, |b| < 1$. There are many identities in the lost notebook, whose proofs naturally use Heine's transformation [5, Chapter 1]. One consequence of (10.1) is, for $|aq| < 1$ [63, p. 38],

$$\sum_{n=0}^{\infty} \frac{(-aq)^n}{(-aq^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-aq; q)_n}. \tag{10.2}$$

To see how (10.1) can be used to prove (10.2), consult [5, p. 25, Entry 1.6.4]. For partition-theoretic proofs, see [33] and [27].

Our second identity is given by [63, p. 35], [5, p. 35, Entry 1.7.9]

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_n (1 - q^{2n+1})} = f(q^3, q^5), \tag{10.3}$$

where $f(a, b)$ is defined by (7.2). We see partitions at work on the left-hand side of (10.3), but on the right-hand side, we observe that these partitions have cancelled each other out, except on a thin set of quadratic exponents. For a combinatorial proof of (10.3), see [27].

11. The Rogers–Ramanujan Continued Fraction

As the name suggests, the Rogers–Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1} \, + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots, \qquad |q| < 1, \tag{11.1}$$

was first studied by L. J. Rogers [67] in 1894 and then by Ramanujan before he departed for England, for his notebooks [62] contain many properties of this continued fraction. His study of $R(q)$ continued unabatedly in the lost notebook, with the first five chapters of [4] focusing on $R(q)$. Readers may find it strange that the first numerator of $R(q)$ in (11.1) is $q^{1/5}$. Because $R(q)$ lives in the homes of theta functions and modular forms, once we see how it relates to the other functions living in the same homes, we understand more fully why $q^{1/5}$ is a part of the continued fraction.

In his first letter to Hardy [61, p. xxvii], [31, p. 29], Ramanujan offered the elegant values of

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

and $R(-e^{-\pi})$, and in his second letter, he communicated the value of $R(e^{-2\pi\sqrt{5}})$ [61, p. xxviii], [31, p. 37]. In his lost notebook, Ramanujan recorded several further values; e.g., on page 46 of [63],

$$R(-e^{-\pi\sqrt{3}}) = \frac{(3+\sqrt{5}) - \sqrt{6(5+\sqrt{5})}}{4}.$$

A key to evaluating $R(q)$ in closed form is the identity (found in Ramanujan's notebooks [16, p. 267])

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}, \tag{11.2}$$

where $f(-q)$ is defined by (7.4), and a similar identity involving $R^5(q)$ that can be derived from (11.2). If one can evaluate the quotient of Dedekind eta functions on the right-hand side of (11.2) for a certain value of q, then one can determine $R(q)$ by solving a simple quadratic equation. The evaluation of quotients of eta functions at points $e^{-\pi\sqrt{n}}$ is usually quite difficult. To this end, evaluating appropriate *class invariants*, which are certain quotients of eta functions at $e^{-\pi\sqrt{n}}$, is often helpful. The first general theorem for determining values of $R(q)$ in this direction was developed by the author, H. H. Chan, and L.–C. Zhang [25].

The function $R(q)$ satisfies several beautiful modular equations. For example [4, p. 92], if $u = R(q)$ and $v = R(q^2)$, then

$$\frac{v-u^2}{v+u^2} = uv^2.$$

Because of limitations of space, we must desist from providing further properties for $R(q)$, but in closing we remark that $R(q)$ also serves as a model, in that its properties guided Ramanujan and others that followed in their quests of finding analogous theorems for other q-continued fractions. In particular, see Chapters 6–8 in [4].

12. Eisenstein Series

Continuing in his lost notebook the study of Eisenstein series made in [59], [61, pp. 136–162], Ramanujan offers many further discoveries about these series. Furthermore, published with his lost notebook are several letters that Ramanujan wrote to Hardy from nursing homes during his last two years in England; these letters are primarily devoted to Eisenstein series. An account of all of these discoveries can be found in the last six chapters in [5]. In this section, we briefly discuss some of these results.

In Ramanujan's notation, the three primary Eisenstein series are

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k},$$

$$Q(q) := 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k},$$

$$R(q) := 1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}.$$

For $q = \exp(2\pi i\tau)$, $\operatorname{Im}\tau > 0$, in more contemporary notation, $Q(q) = E_4(\tau)$ and $R(q) = E_6(\tau)$. These two functions are modular forms of weights 4 and 6, respectively, and are the "building blocks" for modular forms on the full modular group. The function $P(q)$ is not a modular form, but is a quasi-modular form, because it satisfies the same transformation formulas as an ordinary modular form.

As emphasized in Section 9, Hardy and Ramanujan [50], [61, pp. 276–309] found an asymptotic series for the partition function $p(n)$, which arises from the power series coefficients of the reciprocal of the Dedekind eta-function. As they indicated near the end of their paper, their methods also apply to coefficients of further modular forms that are analytic in the upper half-plane. In their last jointly published paper [51], [61, pp. 310–321], they considered a similar problem for the coefficients of modular forms having a simple pole in a fundamental region, and, in particular, they applied their theorem to find interesting series representations for the coefficients of the reciprocal of the Eisenstein series $E_6(\tau)$. In letters from nursing homes, Ramanujan calculated formulas for the coefficients of further quotients of Eisenstein series. The formulas, which were first proved in papers by the author with P. Bialek [21] and with Bialek and A. J. Yee [22], do not fall under the purview of the general theorem from [51]. As they are too complicated to offer in a short survey, we invite readers to examine them in the aforementioned papers or to consult [5, Chapter 11].

Ramanujan recorded many beautiful identities for Eisenstein series in [63]. We close this section with one of them [5, p. 331]. If $f(-q)$ is defined by (7.4), then

$$Q(q) = \frac{f^{10}(-q)}{f^2(-q^5)} + 250qf^4(-q)f^4(-q^5) + 3125q^2\frac{f^{10}(-q^5)}{f^2(-q)}.$$

13. The Circle and Divisor Problems

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. The famous *circle problem* is to determine the precise order of magnitude

for the "error term" $P(x)$ defined by

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + P(x), \tag{13.1}$$

as $x \to \infty$, where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted. We now explain why this problem is called the *circle problem*. Each representation of n as a sum of two squares can be associated with a lattice point in the plane. For example, $5 = (-2)^2 + 1^2$ can be associated with the lattice point $(-2, 1)$. Then each lattice point can be associated with a unit square, say that unit square for which the lattice point is in the southwest corner. Thus, the sum in (13.1) is equal to the number of lattice points in the circle of radius $\sqrt{x}$ centered at the origin, or to the sum of the areas of the aforementioned squares, and the area of this circle, namely πx, is a reasonable approximation to the sum of the areas of these rectangles. Gauss showed quite easily that the error made in this approximation is $P(x) = O(\sqrt{x})$.

In 1915, Hardy [49] proved that $P(x) \ne O(x^{1/4})$, as x tends to ∞. In other words, there is a sequence of points $\{x_n\}$ tending to ∞ on which $P(x_n) \ne O(x_n^{1/4})$. (He actually proved a slightly stronger result.) In connection with his work on the *circle problem*, Hardy [49] proved that

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{13.2}$$

where $J_1(x)$ is the ordinary Bessel function of order 1. After Gauss, almost all efforts toward obtaining an upper bound for $P(x)$ have ultimately rested upon (13.2), and methods of estimating the approximating trigonometric series that is obtained from the asymptotic formula for $J_1(2\pi\sqrt{nx})$ as $n \to \infty$. In 1906, W. Sierpinski [68] proved that $P(x) = O(x^{1/3})$ as x tends to ∞, and there have been many improvements in a century of work since then, with the best current result being $P(x) = O(x^{131/416+\epsilon})$, for every $\epsilon > 0$, due to M. N. Huxley [55] ($\frac{131}{416} = 0.3149\ldots$). It is conjectured that $P(x) = O(x^{1/4+\epsilon})$, for every $\epsilon > 0$. In a footnote, Hardy remarks, "The form of this equation was suggested to me by Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1) + d(2) + \cdots + d(n)$, where $d(n)$ is the number of divisors of n." In this same paper, Hardy relates a beautiful identity of Ramanujan connected with $r_2(n)$, namely, for $a, b > 0$, [49, p. 283],

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}},$$

which is not given elsewhere in any of Ramanujan's published or unpublished work. These facts indicate that Ramanujan and Hardy undoubtedly had probing conversations about the *circle problem*.

On page 335 in [63], which is *not* in Ramanujan's lost notebook, but which is in one of those fragments published *with* the lost notebook, Ramanujan offers two identities

involving Bessel functions. To state Ramanujan's claims, we need to first define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{13.3}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

Entry 13.1 (p. 335)**.** *If* $0 < \theta < 1$ *and* $x > 0$*, then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x\left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{13.4}$$

Entry 13.1 was first proved by the author, S. Kim, and A. Zaharescu [29]. In [28], the same three authors proved (13.4) with the product mn of the summation indices tending to ∞. Entry 13.1 was first established with the order of summation reversed from that prescribed by Ramanujan in (13.4) [34].

The Bessel functions in (13.4) bear a striking resemblance to those in (13.2), and so it is natural to ask if there is a connection between the two formulas. Berndt, Kim, and Zaharescu [28] proved the following corollary, as a consequence of their reinterpreted meaning of the double sum. It had been previously established (although not rigorously) by Berndt and Zaharescu in [34] as a corollary of their theorem on twisted divisor sums arising from Entry 13.1.

Corollary 13.2. *For any* $x > 0$*,*

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}. \tag{13.5}$$

Can Corollary 13.2 be employed in place of (13.2) to effect an improvement in the error term for the circle problem? The advantage of (13.5) is that $r_2(n)$ does not appear on the right-hand side; the disadvantage is that one needs to estimate a double sum, instead of a single sum in (13.2).

The second identity on page 335 pertains to the famous *Dirichlet divisor problem*. Let $d(n)$ denote the number of positive divisors of the integer n. Define the "error term" $\Delta(x)$, for $x > 0$, by

$$\sum_{n\le x}{}' d(n) = x\left(\log x + (2\gamma - 1)\right) + \frac{1}{4} + \Delta(x), \tag{13.6}$$

where γ denotes Euler's constant, and where the prime $'$ on the summation sign on the left side indicates that if x is an integer, then only $\frac{1}{2}d(x)$ is counted. The famous *Dirichlet divisor problem* asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$. Counting lattice points under a hyperbola, Dirichlet easily showed (13.6) with $\Delta(x) =$

$O(\sqrt{x})$. Estimates that have been established for the error term $\Delta(x)$ are similar to those that have been proved for $P(x)$. G. F. Voronoï [69] established a representation for $\Delta(x)$ in terms of Bessel functions with his famous formula

$$\sum_{n\le x}{}' d(n) = x\left(\log x + (2\gamma - 1)\right) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{13.7}$$

where $x > 0$ and

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z). \tag{13.8}$$

Here $Y_1(x)$ denotes the Bessel function of the second kind of order 1, and $K_1(x)$ denotes the modified Bessel function of order 1. Since its appearance in 1904, (13.7) has been the starting point for nearly all attempts at finding an upper bound for $\Delta(x)$.

We close this section by stating the second identity on page 335 of [63]. Note that the Bessel functions that appear below are the same as those in (13.7). Proofs of Entry 13.3 with the order of summation reversed from that given by Ramanujan in (13.9) and with the product of the summation indices m and n approaching ∞ can be found in the author's paper with Kim and Zaharescu [28]. See also [7, Chapter 2].

Entry 13.3 (p. 335)**.** *Let $F(x)$ be defined by* (13.3)*, and let $I_1(x)$ be defined by* (13.8)*. Then, for $x > 0$ and $0 < \theta < 1$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{13.9}$$

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Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: berndt@illinois.edu

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A Formula of S. Ramanujan

R. SITARAMACHANDRARAO*

Department of Mathematics, University of Toledo, Toledo, Ohio 43606

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IN MEMORY OF S. RAMANUJAN

In this paper, we discuss various equivalent formulations for the sum of an infinite series considered by S. Ramanujan. In the process, we also evaluate, in closed form, various classes of related infinite series. © 1987 Academic Press, Inc.

1. INTRODUCTION

In Chapter 9, Volume 2 of his notebooks, Ramanujan (cf. [19, formula (11.3)] stated that

$$G(1) := \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right)$$
$$= \frac{\pi}{4}\sum_{r=0}^{\infty}\frac{(-1)^r}{(4r+1)^3} - \frac{\pi}{3\sqrt{3}}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^3}. \tag{1.1}$$

Recently, Berndt and Joshi (cf. [3, p. 31]), while editing this chapter, observed that this formula is incorrect; in fact, on taking the first three terms of the series defining $G(1)$, they found that $G(1) > 0.1529320988...$ whereas the extreme right of (1.1) is less than 0.1442780636.

In an attempt to obtain a corrected version of this formula, we are led to consider the following 12 classes of infinite series:

$$H_1(s) = \Sigma' r^{-s}k^{-1},$$
$$H_2(s) = \Sigma' r^{-s}(k+r)^{-1},$$
$$H_3(s) = \Sigma'' r^{-s}k^{-1},$$

* On leave from Andhra University, Waltair, India.

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$$H_4(s)=\Sigma' r^{-s}(-1)^{k-1}k^{-1},$$
$$H_5(s)=\Sigma' r^{-s}(-1)^{k-1}(r+k)^{-1},$$
$$H_6(s)=\Sigma'' r^{-s}(-1)^{k-1}k^{-1},$$
$$H_7(s)=\Sigma' (-1)^{r-1}r^{-s}k^{-1},$$
$$H_8(s)=\Sigma' (-1)^{r-1}r^{-s}(r+k)^{-1},$$
$$H_9(s)=\Sigma'' (-1)^{r-1}r^{-s}k^{-1},$$
$$H_{10}(s)=\Sigma' (-1)^{r-1}r^{-s}(-1)^{k-1}k^{-1},$$
$$H_{11}(s)=\Sigma' (-1)^{r-1}r^{-s}(-1)^{k-1}(r+k)^{-1},$$
$$H_{12}(s)=\Sigma'' (-1)^{r-1}r^{-s}(-1)^{k-1}k^{-1}$$

where Σ' ad Σ'' respectively denote

$$\sum_{r=1}^{\infty}\sum_{k=1}^{r} \quad \text{and} \quad \sum_{r=1}^{\infty}\sum_{k=1}^{2r-1}$$

and s denotes a complex number with $\mathrm{Re}\, s>1$ in $H_1(s)$ through $H_6(s)$ and $\mathrm{Re}\, s>0$ in $H_7(s)$ through $H_{12}(s)$.

It may be noted that it is possible to shorten the above notation as follows: For $i,j\in\{0,1\}$, let

$$H_1^{(i,j)}(s)=\Sigma' (-1)^{i(r-1)}r^{-s}(-1)^{j(k-1)}k^{-1},$$
$$H_2^{(i,j)}(s)=\Sigma' (-1)^{i(r-1)}r^{-s}(-1)^{j(k-1)}(r+k)^{-1},$$
$$H_3^{(i,j)}(s)=\Sigma'' (-1)^{i(r-1)}r^{-s}(-1)^{j(k-1)}k^{-1}.$$

Then we note that for $i,j\in\{0,1\}$ and $k\in\{1,2,3\}$

$$H_{6i+3j+k}(s)=H_k^{(i,j)}(s).$$

Some of these infinite series have been studied, for positive integer values of s, by several authors and we refer to Euler [7, 8], Nielsen [18], Ramanujan [19], Rutledge and Douglass [20], Klamkin [13, 14], Williams [27], Briggs, Chowla, Kempner, and Mientka [4], Lehner and Newman [15], Gupta [10], Jordan [11], Kanemitsu [12], the author and Sivaramasarma [21, 22], Sivaramasarma [25], the author and Subbarao [23, 24], Bruckman [5], Georghiou and Philippou [9], Dixon and O'Cinneide [6], Matsuoka [17] and Apostol and Vu [2].

In Section 2 of this paper, we prove several relationships between $H_1(s)$ through $H_{12}(s)$. In Section 3, we discuss the evaluation of each of these infinite series for certain positive integer values of s. In particular, we give a

new proof of the following identity due to Nielsen [18] which apparently goes back to Euler (cf. [18, footnotes on p. 47]): For integral $s \geqslant 2$,

$$2H_1(s) = (s+2)\,\zeta(s+1) - \sum_{i=1}^{s-2} \zeta(s-i)\,\zeta(i+1)$$

where $\zeta(s)$ denotes the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ in $\operatorname{Re} s > 1$ and its analytic continuation in $\operatorname{Re} s > 0$. This result and certain of its special cases were rediscovered by several authors (See Remark 3.1). We also evaluate, in closed form, $H_2(2b)$, $H_3(2b)$, $H_4(s)$, $H_6(2b)$, $H_{10}(2b)$, and $H_{11}(2b)$ where $s \geqslant 2$ and $b \geqslant 1$ are integers. We believe that all these results are new.

Our results of Section 3 give, in particular, the values of $H_1(2)$, $H_2(2)$, $H_3(2)$, $H_4(2)$, $H_6(2)$, $H_7(2)$, $H_{10}(2)$, and $H_{11}(2)$ explicitly in terms of $\zeta(2)$ and $\zeta(3)$. In Section 4, we complete this list by evaluating $H_5(2)$, $H_8(2)$, $H_9(2)$, and $H_{12}(2)$. These evaluations also involve Catalan's constant (see (4.13)).

In Section 5 we prove that

$$\begin{aligned} G(1) &= \tfrac{5}{64}\zeta(4) + \tfrac{1}{8}H_2(3) \\ &= -\tfrac{1}{64}\zeta(4) + \tfrac{1}{8}H_3(3) \\ &= \tfrac{1}{64}\zeta(4) + \tfrac{1}{8}H_6(3) \\ &= \tfrac{35}{64}\zeta(4) - \tfrac{1}{2}H_7(3) \\ &= \tfrac{5}{128}\zeta(4) + \tfrac{7}{32}\zeta(3)\log 2 - \tfrac{1}{8}H_{11}(3) \\ &= \tfrac{3}{16}\zeta(4) - \tfrac{1}{4}A_4 \end{aligned}$$

and

$$G(1) \simeq 0.16227.$$

We also note that

$$\begin{aligned} G(1) &= \tfrac{29}{64}\zeta(4) - \tfrac{7}{8}\zeta(3)\log 2 + \tfrac{1}{2}H_{10}(3) \\ &= -\tfrac{53}{64}\zeta(4) + \tfrac{7}{8}\zeta(3)\log 2 - \frac{\pi^2}{24}(\log 2)^2 + \frac{(\log 2)^4}{12} + Li_4(1/2). \end{aligned}$$

In the above A_4 is the constant defined by (cf. [20, p. 30])

$$A_4 = \sum_{r=2}^{\infty} \frac{(-1)^r}{r^2} \sum_{k=1}^{r-1} \frac{1}{k^2} \tag{1.2}$$

and $Li_4(z)$ is the polylogarithm of order 4 defined by [16]

$$Li_4(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^4},\qquad |z|\leqslant 1.$$

Unfortunately, we could not evaluate, in closed form, any one of $H_2(3)$, $H_3(3)$, $H_6(3)$, $H_7(3)$, $H_{10}(3)$, $H_{11}(3)$, A_4, and $Li_4(\frac{1}{2})$.

2. Relationships between $H_1(s)$ through $H_{12}(s)$

We write $\sigma(s)=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ for $\operatorname{Re} s>0$ so that $\sigma(s)=(1-2^{1-s})\zeta(s)$ for all s in $\operatorname{Re} s>0$, $s\neq 1$ and $\sigma(1)=\log 2$.

The following Lemma is basic to our work.

Lemma 2.1. *For* $\operatorname{Re} s>1$, *we have*

$$H_1(s)+H_2(s)=H_3(s)+\tfrac{1}{2}\zeta(s+1), \tag{2.1}$$

$$H_7(s)+H_8(s)=H_9(s)+\tfrac{1}{2}\sigma(s+1), \tag{2.2}$$

$$H_6(s)+H_{11}(s)=H_4(s)+\tfrac{1}{2}\zeta(s+1), \tag{2.3}$$

$$H_5(s)+H_{12}(s)=H_{10}(s)+\tfrac{1}{2}\sigma(s+1), \tag{2.4}$$

$$H_1(s)-H_7(s)=2^{1-s}H_3(s)+2^{-s}\zeta(s+1), \tag{2.5}$$

$$H_6(s)=H_2(s)+\tfrac{1}{2}\zeta(s+1), \tag{2.6}$$

$$H_{12}(s)=H_8(s)+\tfrac{1}{2}\sigma(s+1), \tag{2.7}$$

$$H_5(s)+H_8(s)=H_{10}(s), \tag{2.8}$$

$$H_2(s)+H_{11}(s)=H_4(s). \tag{2.9}$$

Proof. The results (2.1) through (2.5) follow directly from the definitions and the results (2.6) through (2.9) follow from the easily proved identity

$$\sum_{k=1}^{2r-1}\frac{(-1)^{k-1}}{k}=\sum_{k=1}^{r}\frac{1}{r+k}+\frac{1}{2r}.$$

3. Evaluation of $H_1(s)$ through $H_{12}(s)$ for Certain Values of s

Throughout this section, $a\geqslant 2$ and $b\geqslant 1$ denote integers and $\zeta(s)=(2^s-1)\zeta(s)$ for $\operatorname{Re} s>1$.

Theorem 3.1. $2H_1(a)=(a+2)\zeta(a+1)-\sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1)$.

Remark 3.1. Theorem 3.1 has a long history and evidently Euler (cf. [7] and [8], p. 228) was the first to discover it in case $a=2$. Nielsen (cf. [18, pp. 37–51]) developed a systematic method of proving Theorem 3.1 and several related results. In case $a=2$, Theorem 3.1 was rediscovered by Ramanujan (cf. [19, Chap. 9, entry 9]; [3, p. 25]), Briggs *et al.* [4], Klamkin [13], and Bruckman [5]. Theorem 3.1 in case $a=3$ was rediscovered in a slightly different form, by Rutledge and Douglass [20] (who ascribe to F. T. Morley and whose paper is not available to the author) and it also appears as a problem proposed by Klamkin [14]. In 1953, Williams [27] and in 1983, Georghiou and Philippou [9] rediscovered Theorem 3.1 and in 1979, the author and Sivaramasarma [21] gave a new proof based on Apostol's extension of a transformation formula due to Lehner and Newman (cf. [1, p. 111]). Also the author and Sivaramasarma [21] obtained a result equivalent to Theorem 3.1. Matsuoka [17] and Apostol and Vu [2] independently discussed the nature of the analytic continuation of $H_1(s)$. In addition to this, Apostol and Vu's paper [2] contains several interesting and related results. Finally certain power series associated with the series defining $H_1(s)$ were investigated by Ramanujan (cf. [19, Chap. 9]) and for proofs of these assertions we refer to Berndt and Joshi (cf. [3, pp. 25–35]).

Proof of Theorem 3.1. We give yet another proof based on

LEMMA 3.1. *Let* $\sum_{r=1}^{\infty} f(r)$ *and* $\sum_{k=1}^{\infty} g(k)$ *be two absolutely convergent series of complex terms. Then*

$$\sum_{r=1}^{\infty} f(r) \sum_{k=1}^{r} g(k) + \sum_{r=1}^{\infty} g(r) \sum_{k=1}^{r} f(k) = \left(\sum_{r=1}^{\infty} f(r) \right) \left(\sum_{k=1}^{\infty} g(k) \right) + \sum_{r=1}^{\infty} f(r)\, g(r). \tag{3.1}$$

In particular, if for complex u, v *with* $\operatorname{Re} u > 1$, $\operatorname{Re} v > 1$

$$P(u, v) = \sum_{r=1}^{\infty} \frac{1}{r^u} \sum_{k=1}^{r} \frac{1}{k^v} \tag{3.2}$$

$$R(u, v) = \sum_{r,\, k=1}^{\infty} \frac{1}{r^u (r+k)^v}, \tag{3.3}$$

then

$$P(u, v) + P(v, u) = \zeta(u)\, \zeta(v) + \zeta(u+v) \tag{3.4}$$

$$R(u, v) + R(v, u) = \zeta(u)\, \zeta(v) - \zeta(u+v). \tag{3.5}$$

Proof. Since $\sum_{r=1}^{\infty} f(r)$ and $\sum_{k=1}^{\infty} g(k)$ converge absolutely, we have

$$\left(\sum_{r=1}^{\infty} f(r)\right)\left(\sum_{k=1}^{\infty} g(k)\right) = \sum_{r,k=1}^{\infty} f(r)\, g(k) = \left(\sum_{\substack{r,k=1 \\ k \leqslant r}}^{\infty} + \sum_{\substack{r,k=1 \\ r \leqslant k}}^{\infty} - \sum_{\substack{r,k=1 \\ r=k}}^{\infty}\right) f(r)\, g(k)$$

from which (3.1) follows.

On taking $f(r) = r^{-u}$ and $g(k) = k^{-v}$ in (3.1), we obtain (3.4). Further, since

$$P(u, v) + R(u, v) = \zeta(u)\, \zeta(v),$$

(3.4) and (3.5) are in fact equivalent.

Now to prove Theorem 3.1, we write $S_r = 1 + 1/2 + \cdots + 1/r$ so that

$$S_r = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{r+k}\right) = r \sum_{k=1}^{\infty} \frac{1}{k(r+k)}.$$

Hence we have

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{S_r}{r^a} &= \sum_{r=1}^{\infty} \frac{1}{r^{a-1}} \sum_{k=1}^{\infty} \frac{1}{k(r+k)} \\
&= \sum_{r,k=1}^{\infty} \frac{r+k}{r^{a-1}k(r+k)^2} \\
&= \sum_{r,k=1}^{\infty} \frac{1}{r^{a-2}k(r+k)^2} + R(a-1, 2) \\
&= \sum_{r,k=1}^{\infty} \frac{r+k}{r^{a-2}k(r+k)^3} + R(a-1, 2) \\
&= \sum_{r,k=1}^{\infty} \frac{1}{r^{a-3}k(r+k)^3} + R(a-2, 3) + R(a-1, 2).
\end{aligned}$$

Continuing this procedure, we have by (3.5)

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{S_r}{r^a} &= \sum_{r,k=1}^{\infty} \frac{1}{rk(r+k)^{a-1}} + \sum_{i=2}^{a-1} R(i, a+1-i) \\
&= \sum_{r,k=1}^{\infty} \frac{r+k}{rk(r+k)^a} \\
&\quad + \frac{1}{2} \sum_{i=2}^{a-1} \{R(i, a+1-i) + R(a+1-i, i)\} \\
&= 2 \sum_{r,k=1}^{\infty} \frac{1}{r(r+k)^a} + \frac{1}{2} \sum_{i=2}^{a-1} \{\zeta(i)\, \zeta(a+1-i) - \zeta(a+1)\}.
\end{aligned}$$

Since

$$\sum_{r,k=1}^{\infty} \frac{1}{r(r+k)^a} = \sum_{n=2}^{\infty} \frac{1}{n^a} \sum_{r<n} \frac{1}{r} = \sum_{n=2}^{\infty} \frac{1}{n^a} \left(\sum_{r \leqslant n} \frac{1}{r} - \frac{1}{n} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^a} \left(\sum_{r \leqslant n} \frac{1}{r} - \frac{1}{n} \right) = H_1(a) - \zeta(a+1),$$

we find

$$H_1(a) = 2\{H_1(a) - \zeta(a+1)\} + \tfrac{1}{2} \sum_{i=2}^{a-1} \{\zeta(i)\,\zeta(a+1-i) - \zeta(a+1)\}$$

from which Theorem 3.1 follows.

Remark 3.2. The reciprocity relation (3.2) is of frequent occurance in the literature (cf. [18, 21, 23, 2]) and apparently goes back to Euler.

THEOREM 3.2. $2H_1(2a-1) = \sum_{i=2}^{2a-2} (-1)^i \zeta(i)\,\zeta(2a-i)$.

Proof. We need the following extension due to Apostol (cf. [1, p. 111]; [21]) of a transformation formula due to Lehner and Newman [15].

LEMMA 3.2. *Let $f(x, y)$ be a complex valued function defined for positive integral x and y. Then*

$$\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} f(r,k) = \sum_{r=1}^{n} f(r,r) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} \{f(k,r) + f(r,k) - f(k,r-k)\}.$$

Proof. For a lattice point theoretic proof of the lemma, we refer to the author and Sivaramasarma [21]. Here we give a simpler proof. We have

$$\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} = \sum_{1 \leqslant r,k \leqslant n} f(r,k) - \sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \leqslant n}} f(r,k)$$

$$= \left(\sum_{1 \leqslant r < k \leqslant n} + \sum_{1 \leqslant k < r \leqslant n} + \sum_{\substack{1 \leqslant r,k \leqslant n \\ r=k}} \right) f(r,k) - \sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \leqslant n}} f(r,k)$$

$$= \sum_{k=2}^{n} \sum_{r=1}^{k-1} f(r,k) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} f(r,k) + \sum_{r=1}^{n} f(r,r)$$

$$- \sum_{u=2}^{n} \sum_{r+k=u} f(r,k)$$

$$= \sum_{r=1}^{n} f(r,r) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} (f(r,k) + f(k,r)) - \sum_{u=2}^{n} \sum_{k=1}^{u-1} f(u-k,k)$$

and the lemma follows.

To prove Theorem 3.2, we take $f(x, y) = 1/x^{2a-1}(x+y)$ in Lemma 3.2 and obtain

$$\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} \frac{1}{r^{2a-1}(r+k)} = \frac{1}{2}\sum_{r=1}^{n} \frac{1}{r^{2a}} + \sum_{r=2}^{n}\sum_{k=1}^{r-1} \left\{ \frac{1}{r^{2a-1}(r+k)} + \frac{1}{k^{2a-1}(r+k)} - \frac{1}{k^{2a-1}r} \right\}$$

$$= \frac{1}{2}\sum_{r=1}^{n} \frac{1}{r^{2a}} + \sum_{r=2}^{n}\sum_{k=1}^{r-1} \left\{ \frac{k^{2a-2} - r^{2a-2}}{r^{2a-1}k^{2a-1}(r+k)} \right\}. \tag{3.6}$$

For $a \geqslant 3$, we have

$$\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} \frac{1}{r^{a-1}(r+k)} \leqslant \sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} \frac{1}{r^2(r+k)} \leqslant \frac{1}{n+1} \sum_{1 \leqslant r \leqslant n} \frac{1}{r^2} \sum_{n+1-r \leqslant k \leqslant n} 1$$

$$= \frac{1}{n+1} \sum_{1 \leqslant r \leqslant n} \frac{1}{r} \leqslant \frac{1+\log n}{n}.$$

Hence from (3.6) we find, on letting n tend to infinity,

$$\frac{1}{2}\zeta(2a) = \sum_{r=2}^{\infty}\sum_{k=1}^{r-1} \frac{r^{2a-2} - k^{2a-2}}{r^{2a-1}k^{2a-2}(r+k)}$$

$$= \sum_{r=2}^{\infty}\sum_{k=1}^{r} \frac{r^{2a-2} - k^{2a-2}}{r^{2a-1}k^{2a-2}(r+k)} = \sum_{r=1}^{\infty}\sum_{k=1}^{r} \frac{r^{2a-2} - k^{2a-2}}{r^{2a-1}k^{2a-2}(r+k)}$$

$$= \sum_{r=1}^{\infty}\sum_{k=1}^{r} \frac{r^{2a-3} - r^{2a-4}k + r^{2a-5}k^2 - \cdots + rk^{2a-4} - k^{2a-3}}{r^{2a-1}k^{2a-2}}$$

$$= \sum_{r=1}^{\infty}\sum_{k=1}^{r} \left\{ \frac{1}{r^2k^{2a-2}} - \frac{1}{r^3k^{2a-3}} + - + \frac{1}{r^{2a-2}k^2} - \frac{1}{r^{2a-1}k} \right\}$$

so that by (3.4),

$$\zeta(2a) + 2H_1(2a-1) = \sum_{r=1}^{\infty}\sum_{k=1}^{r} \{(r^{-2}k^{-2a+2} + r^{-2a+2}k^{-2})$$

$$- (r^{-3}k^{-2a+3} + r^{-2a+3}k^{-3})$$

$$+ \cdots + (r^{-2a+2}k^{-2} + r^{-2}k^{-2a+2})\}$$

$$
\begin{aligned}
&= \{P(2, 2a-2) + P(2a-2, 2)\} \\
&\quad - \{P(3, 2a-3) + P(2a-3, 3)\} \\
&\quad + \cdots + \{P(2a-2, 2) + P(2, 2a-2)\} \\
&= \zeta(2)\,\zeta(2a-2) - \zeta(3)\,\zeta(2a-3) \\
&\quad + \cdots + \zeta(2a-2)\,\zeta(2) + \zeta(2a)
\end{aligned}
$$

from which the theorem follows for $a \geqslant 3$. Theorem 3.1 covers the case $a = 2$.

THEOREM 3.3. $\sum_{i=1}^{a-1} \zeta(2i)\,\zeta(2a-2i) = (a+\frac{1}{2})\,\zeta(2a)$.

Proof. Follows from Theorems 3.1 and 3.2.

THEOREM 3.4. $2H_1(2a-1) = (a+\frac{1}{2})\,\zeta(2a) - \sum_{i=1}^{a-2} \zeta(i+1)\,\zeta(2a-2i-1)$.

Proof. Follows from Theorems 3.2 and 3.3.

Remark 3.3. Theorems 3.2 through 3.4 appear in Nielsen (cf. [18, pp. 37–51]). Theorem 3.3 was rediscovered by Tornheim (cf. [26, corollary, p. 308]), Williams (cf. [27, Theorem I]), and Georghiou and Philippou [9]. Recently, Sivaramasarma (cf. [25, Chap. III]) gave a new proof of Theorem 3.3. His work also contains various related results.

THEOREM 3.5. $2H_4(a) = 2\zeta(a)\,\sigma(1) - a\zeta(a+1) + 2\sigma(a+1) + \sum_{i=1}^{a} \sigma(i)\,\sigma(a-i+1)$.

Proof. Following Nielsen (cf. [18, p. 47]) we write

$$d_{u,v} = \sum_{r=1}^{\infty} \frac{1}{(r+1)^a} \sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} \qquad \text{for any } u > 1,\ v \geqslant 1, \tag{3.7}$$

$$\gamma_{u,v} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{(r+1)^u} \sum_{k=1}^{r} \frac{1}{k^v} \qquad \text{for any } u \geqslant 1,\ v \geqslant 1. \tag{3.8}$$

Then it is known (cf. [18, p. 47, Eq. 8 and p. 50, Eq. 6]) that

$$d_{u,v} = \zeta(u)\,\sigma(v) - \sigma(u+v) + \gamma_{v,u}, \tag{3.9}$$

$$\sum_{i=1}^{b} \sigma(i)\,\sigma(b-i+1) = b\zeta(b+1) - 2\sigma(b+1) + 2\gamma_{1,b}. \tag{3.10}$$

However, by definition,

$$\begin{aligned} d_{u,1} &= \sum_{r=2}^{\infty} \frac{1}{r^u} \sum_{k=1}^{r-1} \frac{(-1)^{k-1}}{k} \\ &= \sum_{r=2}^{\infty} \frac{1}{r^u} \left(\sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} - \frac{(-1)^{r-1}}{r} \right) \\ &= \sum_{r=1}^{\infty} \frac{1}{r^u} \left(\sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} - \frac{(-1)^{r-1}}{r} \right) \\ &= H_4(u) - \sigma(u+1). \end{aligned} \tag{3.11}$$

Hence by (3.9) and (3.11)

$$\begin{aligned} H_4(a) &= d_{a,1} + \sigma(a+1) \\ &= \zeta(a)\,\sigma(1) + \gamma_{1,a} \end{aligned}$$

and the theorem follows from (3.10).

THEOREM 3.6. *We have*

$$4H_2(2b) = (2^{2b+1} - 2b - 3)\,\zeta(2b+1) + 2\sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) - \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}, \tag{3.12}$$

$$4H_3(2b) = (2^{2b+1} + 2b - 1)\,\zeta(2b+1) - 2\sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) + \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}, \tag{3.13}$$

$$4H_6(2b) = (2^{2b+1} - 2b - 1)\,\zeta(2b+1) + 2\sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) - \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}, \tag{3.14}$$

and

$$\begin{aligned} 4H_{11}(2b) = {} & 4\zeta(2b)\,\sigma(1) + (6 - 2^{2-2b} - 2b)\,\zeta(2b+1) \\ & - \bar{\zeta}(2b+1) + \sum_{i=1}^{2b} \sigma(i)\,\sigma(2b-i+1) \\ & - 2\sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) - \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}. \end{aligned} \tag{3.15}$$

Proof. It is known due to Jordan (cf. [11, Eq. (8a)]) that

$$4\sum_{r=1}^{\infty}\frac{1}{r^{2b}}\sum_{k=1}^{r}\frac{1}{2k-1}=(2^{2b+1}-1)\,\zeta(2b+1)-2\sum_{i=1}^{b-1}\zeta(2i)\,\bar{\zeta}(2b+1-2i). \tag{3.16}$$

But

$$4\sum_{r=1}^{\infty}\frac{1}{r^{2b}}\sum_{k=1}^{r}\frac{1}{2k-1}=4\sum_{r=1}^{\infty}\frac{1}{r^{2b}}\sum_{k=1}^{r}\left(\frac{1}{k}+\frac{1}{r+k}-\frac{1}{2k}\right)=2H_1(2b)+4H_2(2b).$$

Thus (3.12) follows from (3.16) and the above.

To prove (3.13), we have by (2.1)

$$4H_3(2b)=4H_1(2b)+4H_2(2b)-2\zeta(2b+1).$$

Hence Theorem 3.1 and (3.12) yield (3.13).

Formula (3.14) follows from (2.6) and (3.12) while (3.15) follows from (2.9), Theorem 3.5, and (3.12). This completes the proof of the theorem.

Remark 3.4. We believe that Theorems 3.5 and 3.6 are new. (3.12) in case $b=1$ is due to the author and Sivaramasarma (cf. [21, Eq. (1.13)]) who also proved its equivalence with a result due to Gupta [10]. (3.14) in case $b=1$ is due to the author and Subbarao (cf. [24, Eq. (2.16)]) while (3.16) in case $b=1$ is stated by Ramanujan (cf. [19, p. 108]) and proved by the author and Sivaramasarma [22]. Apparently, Jordan [11] was first to prove the general result (3.16).

THEOREM 3.7.

$$2^{2b+1}H_7(2b)=(2^{2b+1}b-2b-1)\,\zeta(2b+1)+2\sum_{i=1}^{b-1}\{\zeta(2i)\,\bar{\zeta}(2b+1-2i)-(2^{2b}-1)\,\zeta(2b-i)\,\zeta(i+1)\}. \tag{3.17}$$

and

$$4^bH_{10}(2b)=(3\cdot 2^{2b}-4)\,\zeta(2b)\,\sigma(1)+\left(b+\frac{1}{2}-b\cdot 2^{2b}\right)\zeta(2b+1)-\sum_{i=1}^{b-1}\{\zeta(2i)\,\bar{\zeta}(2b-i+1)+2\bar{\zeta}(2i)\,\zeta(2b+1-2i)+(1-2^{2b+1})\,\zeta(2b-i)\,\zeta(i+1)\}-2^{2b}\sum_{i=1}^{b}\sigma(i)\,\sigma(2b-i+1). \tag{3.18}$$

Proof. By (2.5), we have

$$H_7(s) = H_1(s) - 2^{1-s}H_3(s) - 2^{-s}\zeta(s+1).$$

Hence (3.17) follows from Theorem 3.1 and (3.13).

Now to prove (3.18), we first recall the following result due to Jordan (cf. [11], eq. (9a), p. 683):

$$4^{b+1}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^{2b}}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right)$$
$$=\overline{\zeta}(2b+1)+4\overline{\zeta}(2b)\,\sigma(1)-2\sum_{i=1}^{b-1}\overline{\zeta}(2i)\,\zeta(2b+1-2i). \tag{3.19}$$

Since

$$4^{b+1}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^{2b}}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right)$$
$$=4^{b+1}\sum_{r=1}^{\infty}\frac{1+(-1)^{r-1}}{2r^{2b}}\sum_{k=1}^{r}\frac{1+(-1)^{k-1}}{2k}$$
$$=4^{b}\sum_{r=1}^{\infty}\sum_{k=1}^{r}\left\{\frac{1}{r^{2b}k}+\frac{(-1)^{r-1}}{r^{2b}k}+\frac{(-1)^{k-1}}{r^{2b}k}+\frac{(-1)^{r-1}(-1)^{k-1}}{r^{2b}k}\right\}$$
$$=4^{b}\{H_1(2b)+H_7(2b)+H_4(2b)+H_{10}(2b)\},$$

(3.18) follows from (3.19), Theorems 3.1, 3.5, and (3.17). This completes the proof of the theorem.

4. The Case $s=2$

Our results in Section 3, in particular, yield the following

THEOREM 4.1.

$$H_1(2) = 2\zeta(3), \tag{4.1}$$
$$H_2(2) = \tfrac{3}{4}\zeta(3), \tag{4.2}$$
$$H_3(2) = \tfrac{9}{4}\zeta(3), \tag{4.3}$$
$$H_4(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \tfrac{1}{4}\zeta(3) \tag{4.4}$$
$$H_6(2) = \tfrac{5}{4}\zeta(3) \tag{4.5}$$
$$H_7(2) = \tfrac{5}{8}\zeta(3) \tag{4.6}$$
$$H_{10}(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \tfrac{5}{8}\zeta(3) \tag{4.7}$$
$$H_{11}(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \zeta(3). \tag{4.8}$$

We complete this list by proving

THEOREM 4.2.

$$H_5(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \pi G + \tfrac{23}{16}\zeta(3), \tag{4.9}$$

$$H_8(2) = \pi G - \tfrac{33}{16}\zeta(3), \tag{4.10}$$

$$H_9(2) = \pi G - \tfrac{29}{16}\zeta(3), \tag{4.11}$$

$$H_{12}(2) = \pi G - \tfrac{27}{16}\zeta(3), \tag{4.12}$$

where G denotes the Catalan's constant defined by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915965594177\ldots. \tag{4.13}$$

Proof. First we prove (4.11). For this we recall a result enunciated by Ramanujan (cf. [19, Chap. 9, entry 21]) and proved by Berndt and Joshi (cf. [3, p. 46]): For $|x| \leqslant \pi/4$

$$\begin{aligned}
&\frac{\tan^2 x}{2^2} - \left(1+\frac{1}{3}\right)\frac{(\tan x)^4}{4^2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{(\tan x)^6}{6^2} - \&\,c\Big) \\
&= \frac{x^2}{2}\log_e \operatorname{Tan} x + x\left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \&\,c\right) \\
&\quad + \frac{1}{2}\left(\frac{\cos 2x}{1^3} + \frac{\cos 6x}{3^3} + \frac{\cos 10x}{5^3} + \&\,c\right) \\
&\quad - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&\,c\right).
\end{aligned} \tag{4.14}$$

For the sake of completeness, we indicate a proof of (4.14). We denote the left- (resp. right-) hand side of (4.14) by $f(x)$ (resp. $g(x)$) and observe that

$$f(0+) = g(0+) \qquad \text{and} \qquad f'(x) = g'(x) = x^2/\sin 2x$$

in view of the well known

$$(\tan^{-1} x)^2 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}x^{2r}}{r} \sum_{k=1}^{r} \frac{1}{2k-1}$$

and

$$\cos x + \frac{1}{3}\cos 3x + \cdots = \frac{1}{2}\log\left(\cot\frac{x}{2}\right).$$

On taking $x=\pi/4$ in (4.14) we find

$$\frac{1}{2^2}\left(1-\frac{1+1/3}{2^2}+\frac{1+1/3+1/5}{3^2}-\&\,c\right)$$
$$=\frac{\pi}{4}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&\,c\right)-\frac{1}{2}\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\&\,c\right). \tag{4.15}$$

Hence on one hand we have

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\sum_{k=1}^{r}\frac{1}{2k-1}=\pi G-\frac{7}{4}\zeta(3) \tag{4.16}$$

and on the other

$$\begin{aligned}&\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\sum_{k=1}^{r}\frac{1}{2k-1}\\&=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\left\{\sum_{k=1}^{2r-1}\frac{1}{k}+\frac{1}{2r}-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2r}\right)\right\}\\&=H_9(2)+\tfrac{1}{2}\sigma(3)-\tfrac{1}{2}H_7(2)\\&=H_9(2)+\tfrac{1}{16}\zeta(3)\end{aligned} \tag{4.17}$$

where in the last step we used (4.6). Now (4.11) follows from (4.16) and (4.17). Also (4.10) follows from (2.2), (4.6), and (4.11); finally (4.9) follows from (2.8), (4.7), and (4.10); (4.12) follows from (2.7) and (4.10). This completes the proof of the theorem.

5. The Case $s=3$ and Ramanujan's $G(1)$

In this section, we relate $G(1)$ explicitly to each of $H_2(3)$, $H_3(3)$, $H_6(3)$, $H_7(3)$, $H_{11}(3)$, and A_4, where A_4 is given by (1.2).

Theorem 5.1. $G(1)=\frac{3}{16}\zeta(4)-\frac{1}{4}A_4$.

Proof. We first note (cf. [20, Eq. (18)]) that

$$\sum_{r=2}^{\infty}\frac{(-1)^r}{r^3}\sum_{k=1}^{r-1}\frac{1}{k}=-\frac{1}{2}A_4+\frac{\pi^4}{576}. \tag{5.1}$$

In fact

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

$$= \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^2 n^2}$$

$$= \sum_{r=2}^{\infty} (-1)^r \sum_{k=1}^{r-1} \frac{1}{k^2(r-k)^2}$$

$$= \sum_{r=2}^{\infty} (-1)^r \sum_{k=1}^{r-1} \left\{ \frac{2}{r^3 k} + \frac{2}{r^3(r-k)} + \frac{1}{r^2 k^2} + \frac{1}{r^2(r-k)^2} \right\}$$

$$= 4 \sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \sum_{k=1}^{r-1} \frac{1}{k} + 2 \sum_{r=2}^{\infty} \frac{(-1)^r}{r^2} \sum_{k=1}^{r-1} \frac{1}{k^2}.$$

Hence

$$\sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \sum_{k=1}^{r-1} \frac{1}{k} = -\frac{1}{2} A_4 + \frac{1}{4} \left(\frac{1}{2} \zeta(2) \right)^2$$

$$= -\frac{1}{2} A_4 + \frac{\pi^4}{576}$$

which is (5.1). We also need

$$H_7(3) = \tfrac{23}{32} \zeta(4) + \tfrac{1}{2} A_4. \tag{5.2}$$

To see this, we have by (5.1)

$$-\frac{1}{2} A_4 + \frac{\pi^4}{576} = \sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \sum_{k=1}^{r-1} \frac{1}{k}$$

$$= \sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \left(\sum_{k=1}^{r} \frac{1}{k} - \frac{1}{r} \right)$$

$$= -\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^3} \sum_{k=1}^{r} \frac{1}{k} + \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^4}$$

$$= -H_7(3) + \sigma(4).$$

Since $\sigma(4) = \frac{7}{8} \zeta(4)$ and $\zeta(4) = \pi^4/90$, we obtain (5.2).

Now consider

$$G(1)=\sum_{r=1}^{\infty}\frac{1}{(2r)^3}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right)$$

$$=\sum_{r=1}^{\infty}\frac{1}{(2r)^3}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2r}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2r}\right)\right\}$$

$$=\sum_{r=1}^{\infty}\frac{1+(-1)^r}{2r^3}\sum_{k=1}^{r}\frac{1}{k}-\frac{1}{16}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{k}$$

$$=\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{k}-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^3}\sum_{k=1}^{r}\frac{1}{k}-\frac{1}{16}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{k}$$

$$=\left(\frac{1}{2}-\frac{1}{16}\right)H_1(3)-\frac{1}{2}H_7(3). \tag{5.3}$$

But by Theorem 3.1 and the fact tht $\zeta^2(2)=\pi^4/36=\frac{5}{2}\zeta(4)$, we have

$$H_1(3)=\tfrac{1}{2}(5\zeta(4)-\zeta^2(2))$$

$$=\tfrac{5}{4}\zeta(4). \tag{5.4}$$

Hence from (5.3) and (5.4)

$$G(1)=\tfrac{35}{64}\zeta(4)-\tfrac{1}{2}H_7(3). \tag{5.5}$$

Thus by (5.2)

$$G(1)=\tfrac{35}{64}\zeta(4)-\tfrac{1}{2}\left(\tfrac{23}{32}\zeta(4)+\tfrac{1}{2}A_4\right)-\tfrac{1}{4}A_4+\tfrac{3}{16}\zeta(4)$$

and Theorem 5.1 follows.

Remark 5.1. It may be noted that (5.5) above relates $G(1)$ explicitly with $H_7(3)$.

THEOREM 5.2. $G(1)\simeq 0.16227\ldots$.

Proof. It is known due to Rutledge and Douglass [20], that the first five digits of A_4 are 0.16265. Hence by Theorem 5.1, we see that

$$G(1)\simeq\tfrac{3}{16}(1.08232)-\tfrac{1}{4}(0.16265)$$

$$=0.16227.$$

THEOREM 5.3. $G(1)=\frac{5}{64}\zeta(4)+\frac{1}{8}H_2(3)$.

Proof. We have

$$\begin{aligned} G(1) &= \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right) \\ &= \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left\{1+\frac{1}{2}+\cdots+\frac{1}{2r}-\left(\frac{1}{2}+\cdots+\frac{1}{2r}\right)\right\} \\ &= \frac{1}{8}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{k}+\frac{1}{8}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{r+k}-\frac{1}{16}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{k} \\ &= \tfrac{1}{16}H_1(3)+\tfrac{1}{8}H_2(3). \end{aligned} \tag{5.6}$$

Hence the theorem follows from (5.4).

COROLLARY 5.1. $H_2(3)=\frac{7}{8}\zeta(4)-2A_4$.

Proof. Follows from Theorems 5.1 and 5.3.

THEOREM 5.4. $G(1)=-\frac{1}{64}\zeta(4)+\frac{1}{8}H_3(3)$.

Proof. From (5.6), (2.1), and (5.4), we have

$$\begin{aligned} G(1) &= \tfrac{1}{16}H_1(3)+\tfrac{1}{8}H_2(3)=\tfrac{1}{8}(H_1(3)+H_2(3))-\tfrac{1}{16}H_1(3) \\ &= \tfrac{1}{8}(H_3(3)+\tfrac{1}{2}\zeta(4))-\tfrac{1}{16}\cdot\tfrac{5}{4}\zeta(4) \\ &= -\tfrac{1}{64}\zeta(4)+\tfrac{1}{8}H_3(3). \end{aligned}$$

COROLLARY 5.2. $H_3(3)=\frac{13}{8}\zeta(4)-2A_4$.

Proof. Follows from Theorems 5.1 and 5.4.

THEOREM 5.5. $G(1)=\frac{1}{64}\zeta(4)+\frac{1}{8}H_6(3)$.

Proof. Follows from Theorem 5.3 and (2.6).

COROLLARY 5.3. $H_6(3)=\frac{11}{8}\zeta(4)-2A_4$.

Proof. Follows from Theorems 5.1 and 5.5.

THEOREM 5.6. $G(1)=\frac{7}{32}\zeta(3)\log 2+\frac{5}{128}\zeta(4)-\frac{1}{8}H_{11}(3)$.

Proof. Follows from Theorems 5.3, (2.9), and Theorem 3.5 with $s=3$.

COROLLARY 5.4. $H_{11}(3)=\frac{7}{4}\zeta(3)\log 2-\frac{19}{16}\zeta(4)+2A_4$.

Proof. Follows from Theorems 5.1 and 5.6.

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AN OVERVIEW OF RAMANUJAN'S NOTEBOOKS

Bruce C. Berndt

Generally acknowledged as India's greatest mathematician, Srinivasa Ramanujan was born on 22 December 1887 in Erode, located in the southern Indian State of Tamil Nadu. He began to focus on mathematics at an early age, and, at the age of about fifteen, borrowed a copy of G. S. Carr's *Synopsis of Pure and Applied Mathematics* [29], which served as his primary source for learning mathematics. Carr was a tutor and compiled this compendium of approximately 4000–5000 results (with very few proofs) to facilitate his tutoring. At the age of sixteen, Ramanujan entered the Government College in Kumbakonam, where he lived for most of his life. By that time, Ramanujan was solely devoted to mathematics and consequently failed his examinations at the end of the first year. He therefore lost his scholarship and, because his family was poor, was forced to terminate his formal education.

At about the time Ramanujan entered college, he began to record his mathematical discoveries in notebooks, although the entries on magic squares in Chapter 1 probably emanate from his school days. Living in poverty with no means of financial support, suffering at times from serious illnesses, and working in isolation, Ramanujan devoted all of his efforts to mathematics and continued to record his discoveries without proofs in notebooks for the next six years.

In 1910 Ramanujan moved to Madras where he obtained temporary financial assistance from R. Ramachandra Rao, and subsequently became a clerk in the Madras Port Trust Office. Several people offered support, and Ramanujan was persuaded to write English mathematicians about his mathematical discoveries. One, G. H. Hardy, responded encouragingly and invited Ramanujan to come to Cambridge to develop his mathematical gifts. After some reluctance, Ramanujan boarded a passenger ship for England on 17 March 1914. At about this time, Ramanujan evidently stopped recording his theorems in notebooks, although a few entries in his third notebook were undoubtedly recorded in England. That Ramanujan no longer concentrated on logging entries in his notebooks is evident from two letters that he wrote friends in Madras during his first year in England [25]. In a letter of 13 November 1914 to his friend R. Krishna Rao, Ramanujan confided, "I have changed my plan of publishing my results. I am not going to publish any of the old results in my notebooks till the war is over." And in a letter of 7 January 1915 to S. M. Subramanian, Ramanujan admitted, "I am doing my work very slowly. My notebook is sleeping in a corner for these four or five months. I am publishing only my present researches as I have not yet proved the results in my notebooks rigorously."

On 24 March 1915, near the end of his first winter in Cambridge, Ramanujan wrote his friend E. Vinayaka Row in Madras, "I was not well till the beginning of this term owing to the weather and consequently I couldn't publish any thing

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for about 5 months." By the end of his third year in England, Ramanujan was critically ill, and, for the next two years, he was confined to sanitariums and nursing homes. After World War I ended in 1919, Ramanujan returned home, but his health continued to deteriorate, and on 26 April 1920 Ramanujan died at the age of 32.

Doctors in both England and India had difficulty diagnosing his illness; tuberculosis and a severe vitamin deficiency were perhaps the two most frequently suggested causes. However, D. A. B. Young [74] has recently made a careful examination of all the records and symptoms of Ramanujan's illness and has convincingly concluded that Ramanujan suffered from hepatic amoebiasis (parasitic infection of the liver). Not only do all of Ramanujan's symptons suggest this disease, but Ramanujan's medical history in India also favors this diagnosis. Amoebiasis is a protozoal infection of the large intestine that gives rise to dysentery. The disease is widespread in India, particularly around coastal cities such as Madras. In 1906 Ramanujan left home to attend Pachaiyappa's College in Madras, where he contracted a severe case of dysentery and had to return home for three months. Unless adequately treated, the infection is permanent, although the patient may go for long periods without exhibiting any symptons. Relapses occur when the host–parasite relationship is disturbed, which likely happened in 1909 when Ramanujan again became acutely ill. This illness is very difficult to diagnose, but once diagnosed, it can be readily cured.

Our description of Ramanujan's life has been necessarily brief. For several years, the standard sources about Ramanujan's life have been the obituaries of P. V. Seshu Aiyar, R. Ramachandra Rao, and Hardy, found in Ramanujan's *Collected Papers* [54], and Chapter 1 of Hardy's book [38]. By far, the most comprehensive biography of Ramanujan has been written by R. Kanigel [40]. The letters from and to Ramanujan are also a source of both mathematical and personal information about Ramanujan, and most of the extant letters have recently been compiled with commentary by R. A. Rankin and the author [25].

After Ramanujan died, Hardy strongly urged that Ramanujan's notebooks be edited and published. By "editing," Hardy meant that each claim made by Ramanujan in his notebooks should be examined. If a theorem is known, sources providing proofs should be cited; if an entry is not known, then an attempt should be made to prove it. Ramanujan, in fact, had left his first notebook with Hardy when he returned to India in 1919, and in 1923 Hardy wrote a paper [36], [37, pp. 505–516] about a chapter on hypergeometric series found in the first notebook. In this paper, Hardy pointed out that Ramanujan had independently discovered most of the important classical results in the subject while also discovering several new theorems as well. It should be remarked here that the second notebook contains two expanded chapters on hypergeometric series, and that further results on hypergeometric series, most of them new, can be found at other scattered places in the second and third notebooks.

Hardy sent the first notebook to the University of Madras where Ramanujan's other notebooks and papers were being preserved. Later, in 1925, T. A. Satagopan at the University of Madras made a handwritten copy of the first notebook which was mailed to Hardy. Plans were being made to publish Ramanujan's collected papers and, possibly, his notebooks and other manuscripts. Thus, on 30 August 1923, the Registrar at the University of Madras, Francis Dewsbury, sent the second notebook (in four parts) to Hardy, and in 1925 further papers were sent to him. The original notebooks were returned to Madras, but the remaining papers evidently

were not. It transpired that Ramanujan's *Collected Papers* [54] were published in 1927, but his notebooks and other manuscripts were not published.

Sometime in the late 1920s, G. N. Watson and B. M. Wilson began the task of editing Ramanujan's notebooks. The second notebook, being a revised, enlarged edition of the first, was their primary focus. Wilson was assigned Chapters 2–14, and Watson was to examine Chapters 15–21. Wilson devoted his efforts to this task until 1935, when he died from an infection at the early age of 38. Watson wrote over 30 papers inspired by the notebooks before his interest evidently waned in the late 1930s. Thus, the project was never completed.

It was not until 1957 that the notebooks were made available to the public when the Tata Institute of Fundamental Research in Bombay published a photocopy edition [55], but no editing was undertaken. The first notebook was published in volume 1, and volume 2 comprises the second and third notebooks.

While residing for a year at the Institute for Advanced Study in Princeton, on a cold winter day in early February, 1974, I was reading two papers by Emil Grosswald [34], [35] in which some formulas from the notebooks were proved. I observed that I could prove these formulas by using a theorem I had proved two years earlier on the modular transformations of a large class of functions including the Dedekind eta–function. I was naturally curious to determine if there were other formulas in the notebooks which I could prove employing my methods. Fortunately, the library at Princeton University has a copy of the Tata Institute's edition, and, indeed, I found a few more formulas of the same sort which I could prove. Eventually I wrote two long papers [7], [8] providing proofs of several formulas from the notebooks and many others of a kindred nature.

All of the aforementioned formulas of Ramanujan can be found in Chapter 14 of his second notebook. However, there were many beautiful formulas involving infinite series in Chapter 14 that I could not prove. So, after the spring semester at the University of Illinois ended in May, 1977, I decided to attempt to find proofs for all 87 formulas in Chapter 14. After working on this project for close to a year, George Andrews, in a visit at the University of Illinois, informed me that Watson and Wilson's efforts in editing the notebooks were preserved in Trinity College Library at Cambridge. The librarian kindly sent me a copy of their notes, which have been enormously helpful. Especially helpful have been Watson's notes on the chapters on modular equations. Thus, since May, 1977, I have devoted all of my research efforts to editing Ramanujan's notebooks. This work, accomplished with the help of several mathematicians, is contained in [9], [10], [11], [12], and [13].

A few remarks should be offered about other unpublished manuscripts left by Ramanujan. The most celebrated and important manuscript, Ramanujan's "lost notebook," was rediscovered by Andrews in 1976 at Trinity College Library, Cambridge. An introduction to the "lost notebook" can be found in Andrews' article [1]. The history of this "notebook," actually a sheaf of 138 pages, is unclear. It probably was sent in 1923 from Madras to Hardy, who kept it in his possession until possibly the late 1930s or early 1940s when he passed it to Watson, who, by that time, no longer had the passionate interest in Ramanujan's work that he had earlier. The "notebook" was found in Watson's papers after his death in 1965. An obituary of Watson was written by R. A. Rankin [57], who, in December, 1968, sent Ramanujan's manuscript along with several papers of Watson to Trinity College Library. The "lost notebook" was published along with other manuscripts and letters by Ramanujan in 1988 [55].

As indicated above, Ramanujan left three notebooks. The first notebook, totalling 351 pages, contains 16 chapters of loosely organized material with the remainder unorganized. In the organized part, which ends on page 263 in the pagination of [55], Ramanujan wrote on only one side of the paper. Shortly thereafter, Ramanujan began to write on both sides of the page and then returned to the unused reverse sides to record additional material, so that only about 20 of the 351 pages are actually blank.

The second notebook is a revised enlargement of the first and was probably composed during the nine months that Ramanujan held a scholarship at the University of Madras prior to his departure for England. This notebook contains 21 chapters, comprising 256 pages, followed by 100 pages of miscellaneous material.

The short third notebook contains 33 pages of unorganized entries.

Hardy estimated that the notebooks contain approximately 3000–4000 results. This estimate appears to be correct; in preparing *Ramanujan's Notebooks, Parts I–V* [9]–[13], we counted 3254 results, although we emphasize that different people will tally such a count in different ways. Each chapter contains approximately 50–150 entries.

Hardy estimated that 2/3 of Ramanujan's work in India was rediscovery. This estimate is definitely too high, for at least 1/2 of the material was new when our editing commenced in 1977.

As indicated above, the notebooks contain almost no proofs. Perhaps there are about 10–20 results for which Ramanujan sketches a proof, often with only one sentence. There are several reasons for the absence of proofs.

1. Ramanujan was probably influenced by the style of Carr's book [29].

2. Like most Indian students in his time, Ramanujan worked primarily on a slate. Paper was very expensive. Thus, after rubbing out his proofs with his sleeve, Ramanujan recorded only the final results in his notebooks.

3. Ramanujan never intended that his notebooks be made available to the mathematical public. They were his personal compilation of what he had discovered. If someone had asked him how to prove a particular result in the notebooks, undoubtedly Ramanujan could supply a proof.

Many have speculated about Ramanujan's methods. Indeed, for many parts of Ramanujan's mathematics, it is difficult to make intelligent guesses about how he argued. In other areas, we can be fairly certain about the nature of many of Ramanujan's arguments, although we may not know the precise details. It should be emphasized, however, that Ramanujan doubtless thought like any other mathematician; he just thought with more insight than most of us. Although Ramanujan, and others among us, may attribute our revelations to divine inspiration, it does not help us to understand Ramanujan's discoveries by claiming that they were obtained by intuition, the inspiration of Goddess Namagiri, or some other mystical means.

Since Ramanujan's notebooks were only intended for his personal use, we might surmise that they contain several errors. Of course, there are occasional misprints. However, perhaps surprisingly, there are very few serious mistakes. Because Ramanujan had little formal training, his proofs were undoubtedly not rigorous in many instances. Despite this, Ramanujan was keenly aware of when his unrigorous thinking yielded correct results, and when it did not. Most of Ramanujan's mistakes arise from his claims in analytic number theory, where his unrigorous methods led him astray. In particular, Ramanujan thought that his approximations and asymp-

totic expansions were considerably more accurate than were warranted. In [12], these shortcomings are discussed in detail. However, a warning should be given to those examining the notebooks for the first time. It is easy to conclude that many formulas are incorrect. Ramanujan often recorded results in unconventional manners, but, properly interpreted, almost always Ramanujan is correct.

Although Ramanujan is primarily known to the mathematical community as a number theorist, only a small portion of the material in the notebooks is devoted to number theory. Most of the contents come under the purview of classical analysis. However, numerous results, e.g., the several hundred theorems on theta functions and modular equations, are at the interface of analysis and number theory. Opening the notebooks, one will likely focus on some infinite series. Infinite series were undoubtedly Ramanujan's first love; perhaps only Euler possessed Ramanujan's talents in working with infinite series.

Next, we briefly discuss several topics examined by Ramanujan in his notebooks. In this brief account, we cannot give a complete description of Ramanujan's important contributions to any of these areas, and indeed many topics will not be discussed at all. We have moderately emphasized Ramanujan's claims in the unorganized portions of the notebooks. To keep the number of references to a manageable number, we ask readers to consult our books [9]–[13], where additional history and fuller lists of references are provided.

1. Elementary Mathematics

Many of Ramanujan's discoveries can be appreciated by those with only a knowledge of high school algebra. Chapter 2 in the second notebook, the unorganized portions of the second and third notebooks [12, Chapter 22], and the problems that Ramanujan submitted to the *Journal of the Indian Mathematical Society* [54] are excellent sources for these gems.

Those familiar with the famous taxicab number 1729 might correctly surmise that Ramanujan enjoyed finding equal sums of powers. Thus, for example, if $a + b + c = 0$ [12, p. 96],

$$2(ab + ac + bc)^4 = a^4(b - c)^4 + b^4(c - a)^4 + c^4(a - b)^4.$$

In fact, in his third notebook [55, p. 385], Ramanujan recorded similar formulas for $2(ab + ac + bc)^{2n}$, for $n = 1$–4, and wrote "and so on" to indicate that he possessed a general procedure for finding such formulas. We think S. Bhargava found Ramanujan's secret here, and we refer the reader to his elegant theorem in [12, p. 97].

One of Ramanujan's most remarkable formulas is the following polynomial identity. Let

$$\begin{aligned} F_{2m}(a, b, c, d) =& (a + b + c)^{2m} + (b + c + d)^{2m} - (c + d + a)^{2m} \\ & - (d + a + b)^{2m} + (a - d)^{2m} - (b - c)^{2m}. \end{aligned}$$

Then [12, p. 102]

$$64F_6(a, b, c, d)F_{10}(a, b, c, d) = 45F_8^2(a, b, c, d),$$

first proved by Bhargava and the author [12, pp. 102–104].

Did you know that [12, pp. 19, 39]

$$2\sin(\pi/18) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \cdots}}}},$$

where the sequence of signs $-, +, +, \ldots$ has period 3, or that

$$(\cos 40^o)^{1/3} + (\cos 80^o)^{1/3} - (\cos 20^o)^{1/3} = \sqrt[3]{\tfrac{3}{2}(\sqrt[3]{9} - 2)}?$$

Ramanujan was fond of stating intriguing formulas such as those above, but as in most instances, they are special cases of more general theorems that he established.

2. Number Theory

We cite only one theorem in the notebooks from the theory of numbers, as reformulated by K. S. Williams [12, p. 71].

Theorem. *Let a, b, A, and B denote positive integers satisfying the conditions*

$$(a, b) = 1 = (A, B), \qquad ab \neq \text{square}.$$

Suppose also that every prime $p \equiv B \pmod{A}$ with $(p, 2ab) = 1$ is expressible in the form $ax^2 - by^2$ for some integers x and y. Then every prime q such that $q \equiv -B \pmod{A}$ and $(q, 2ab) = 1$ is expressible in the form $bX^2 - aY^2$ for some integers X and Y.

As an example, let $a = 1, b = 7, A = 28$, and $B = 9$. If p is a prime such that $p \equiv 9 \pmod{28}$, there exist integers x and y such that $p = x^2 - 7y^2$ [42, Table III]. For example, $37 = 10^2 - 7 \cdot 3^2$. Then by the theorem of Ramanujan and Williams, if q is a prime such that $q \equiv -9 \pmod{28}$, then there exist integers X and Y such that $q = 7X^2 - Y^2$. For example, $19 = 7 \cdot 2^2 - 3^2$.

Many of Ramanujan's theorems on theta functions and modular equations have applications to number theory. In particular, see Sections 10 and 12.

We also remark that a forerunner of the Hardy–Ramanujan "circle method" can be found in the notebooks. Ramanujan attempted (unrigorously) to use the influence of several singularities of the generating function in order to obtain an asymptotic formula [12, pp. 62–66]. Exploiting the function's behavior around many singularities is the crux of the "circle method."

3. Infinite Series

It seems appropriate to begin this section with one of the formulas that Grosswald [34], [35] proved and which fostered our addiction to Ramanujan's formulas. Let $\zeta(s)$ denote the Riemann zeta–function. If $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and n is any nonzero integer, then

$$\begin{aligned}
&\alpha^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k} - 1}\right)\\
&= (-\beta)^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k} - 1}\right)\\
&\quad - 2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k, \qquad (3.1)
\end{aligned}$$

where $B_j, j \geq 0$, denotes the jth Bernoulli number. For a discussion of the many proofs of (3.1), see [10, p. 276]. Curiously, in his first notebook, Ramanujan stated a more general result [12, pp. 429, 430], which he did not record in his second notebook. If $\alpha = \beta = \pi$ and n is odd and positive, (3.1) reduces to

$$\zeta(2n+1) = 2^{2n}\pi^{2n+1} \sum_{k=0}^{n+1} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{2n+2-2k}}{(2n+2-2k)!} - 2 \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\pi k} - 1},$$

which was first discovered by Lerch [43] in 1901. Thus, $\zeta(2n+1)$ equals a rational multiple of π^{2n+1} plus a very rapidly convergent series, i.e., $\zeta(2n+1)/\pi^{2n+1}$ is "almost" rational.

One of Riemann's proofs of the functional equation of $\zeta(s)$ employs the theta transformation formula, which is easily established via the Poisson summation formula, and which was also discovered by Ramanujan [10, p. 253], [11, p. 43]. Quite remarkably, Ramanujan went one step further and found a transformation for doubly exponential series, which he stated in a peculiar fashion in the unorganized portion of his second notebook [55, p. 279]. We offer a more precise version, established in a paper with J. L. Hafner [24]. Let $n, \alpha, \beta > 0$ with $\alpha\beta = 2\pi$. Then

$$\alpha \sum_{k=0}^{\infty} e^{-ne^{k\alpha}} = \alpha \left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} n^k}{k!(e^{k\alpha} - 1)} \right) - \gamma - \log n + 2 \sum_{k=1}^{\infty} \varphi(k\beta),$$

where γ denotes Euler's constant and

$$\varphi(\beta) = \frac{1}{\beta} \mathrm{Im} \left\{ n^{-i\beta} \Gamma(i\beta + 1) \right\}.$$

This can also be proved by means of the Poisson summation formula, but the proof is considerably more intricate than that for the theta transformation formula.

We give one more example. Let $n, \alpha, \beta > 0$ with $\alpha\beta = 2\pi$. Let

$$\psi(n) = \int_0^{\infty} \varphi(x) \cos(nx) dx$$

(for a suitable class of functions φ). Then on page 312 in his second notebook, Ramanujan claims that

$$\frac{\alpha}{2} \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(\alpha/n)}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(\beta/n)}{n}, \tag{3.2}$$

where $\mu(n)$ denotes the Möbius function. The reader need not be told that this "Möbius analogue" of the Poisson summation formula is astounding! Well, (3.2) is too astounding; it is false! However, (3.2) can be corrected by adding a series arising from the complex zeros of $\zeta(s)$. See [13, Chapter 35, Entries 35–37] for further discussion and examples.

The reader will have noticed that the three examples we have chosen have a symmetry in α and β. Indeed, Ramanujan derived many formulas of this type. He also originally summed many infinite series in closed form, derived many beautiful partial fraction expansions, and discovered several summation formulas akin to the

the Abel–Plana summation formula, among his hundreds of beautiful discoveries in the realm of infinite series.

4. Integrals

Although Ramanujan evidently devoted considerably more effort to infinite series than to integrals, several integrals bear his name or are important in current research. Chapter 13 in his second notebook [10] and Chapter 28 in [12], which is a compilation of his results on integrals scattered among the unorganized portions of the second and third notebooks, contain most of the notebooks' entries on integrals, while several of his papers [54] focus on integrals. We cite one example. Let $n \geq 0$, put $v = u^n - u^{n-1}$, and define

$$\varphi(n) = \int_0^1 \frac{\log u}{v} dv.$$

Then, for $n > 0$ [12, p. 326],

$$\varphi(n) + \varphi\left(\frac{1}{n}\right) = \frac{\pi^2}{6}, \tag{4.1}$$

which was proved and generalized by the author and R. J. Evans [22]. Although not readily apparent, (4.1) is related to a reciprocity theorem for the dilogarithm $\mathrm{Li}_2(s)$, defined for any complex number s by

$$\mathrm{Li}_2(s) = -\int_0^s \frac{\log(1-u)}{u} du, \tag{4.2}$$

where the principal branch of $\log w$ is taken. In fact, in Chapter 9 Ramanujan studied the dilogarithm, trilogarithm, and several functions akin to the dilogarithm.

One of Ramanujan's favorite and most powerful integral theorems is his "Master Theorem," which is the subject of his Quarterly Reports, written for the University of Madras, where he held a scholarship for nine months before departing for England. In this theorem Ramanujan asserts that

$$\int_0^\infty x^{n-1} \sum_{k=0}^\infty \frac{\varphi(k)(-x)^k}{k!} dx = \Gamma(n)\varphi(-n). \tag{4.3}$$

Of course, (4.3) needs to be properly interpreted. See Hardy's book [38, pp. 189, 190] and our book [9, pp. 298–331] for conditions of validity, many applications of (4.3) and its converse, and also for several of its variations.

5. Asymptotic Expansions and Approximations

Although Ramanujan is well known for his asymptotic formulas in number theory, in particular, for his remarkable asymptotic series for the partition function $p(n)$ in a joint paper with Hardy [39], [54, pp. 276–309], he has not been recognized for his asymptotic methods and theorems in analysis. The reason is clear; his beautiful asymptotic formulas, both general and specific, laid hidden in his notebooks

for many years. Chapters 3, 13, and 15 in his second notebook [9], [10], and a long chapter in [13, Chapter 36], collecting the many results scattered in the unorganized pages of the second and third notebooks, contain most of Ramanujan's discoveries. We discuss two specific examples.

First, let $a, p > 0$. As p tends to ∞,

$$\sum_{n=0}^{\infty} \frac{(a+n)^{n-1}}{(2p+a+n)^{n+1}} \sim \frac{1}{2ap} - e^{-2p} \sum_{n=0}^{\infty} \frac{(-1)^n P_{2n}(p)}{(a+p)^{2n+1}}, \tag{5.1}$$

where $P_{2n}(p), n \geq 0$, is a polynomial in p of degree $n-1$. In particular,

$$P_0(p) = \frac{1}{2p}, \quad P_2(p) = \frac{1}{6}, \quad P_4(p) = \frac{1}{30} + \frac{p}{6}, \quad \text{and} \qquad P_6(p) = \frac{1}{42} + \frac{p}{6} + \frac{5p^2}{18}.$$

For several related theorems and generalizations of (5.1), see [13] and our paper with Evans [23].

Second, as t tends to $0+$,

$$F(t) := 2\sum_{n=0}^{\infty} (-1)^n \left(\frac{1-t}{1+t}\right)^{n(n+1)} \sim 1 + t + t^2 + 2t^3 + 5t^4 + 17t^5 + \cdots . \tag{5.2}$$

More precisely, as t tends to $0+$,

$$F(t) \sim \left(\frac{1+t}{1-t}\right)^{1/4} \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{2^{2n} n!} \log^n \left(\frac{1+t}{1-t}\right). \tag{5.3}$$

Obviously, (5.2) can be easily obtained from the more complete version (5.3); the latter was not given by Ramanujan. A proof of (5.3) can be found in [13, Chapter 36]. The function $F(t)$ resembles a theta function, but it is not; it is a false theta function in the sense of L. J. Rogers [59].

6. The Gamma Function and Related Functions

The gamma function plays a ubiquitous role in Ramanujan's work, although he did not contribute any significant theorems about the gamma function itself. His integrals involving gamma functions are important in recent research on orthogonal polynomials, and his q–analogues of the gamma and beta functions arise in recent work of R. A. Askey, J. Wilson, and many others on q–orthogonal polynomials. See [10, p. 225], [11, p. 29], and [6] for more specific details and references.

In Chapters 8 and 9 of his second notebook [55], [9], Ramanujan studied several fascinating analogues of the gamma function and derived properties analogous to familiar ones for the gamma function, in particular, Gauss's multiplication formula, Stirling's formula, and Kummer's formula.

In Section 9, we discuss Ramanujan's continued fractions for products of gamma functions.

7. Hypergeometric Functions

We mentioned earlier that Ramanujan not only rediscovered most of the primary classical theorems about hypergeometric series but that he found many new results

as well. First, he found many elegant product formulas for hypergeometric series. Some of these were communicated in his first two letters to Hardy, and after these letters were published with Ramanujan's *Collected Papers* [54] in 1927, W. N. Bailey and others wrote several papers on this subject. Second, Ramanujan discovered some elegant formulas for certain partial sums of hypergeometric series; some, being also found in the aforementioned letters, spawned papers in the late 1920's and early 1930's. Third, and perhaps most importantly, Ramanujan devised various asymptotic expansions for hypergeometric functions. We give one example [10, p. 195] as reformulated by Evans [31, p. 550].

Theorem. *Let $a = c+d$ with $c, d > 0$. Then, as a, c, and d tend to ∞ (equivalently, as $a/(cd) \to 0$),*

$$ {}_2F_1(a, 1; c; c/a) \sim c \left(\frac{a^a \Gamma(c)\Gamma(d)}{2\Gamma(a)c^c d^d} + B_1 \frac{a}{cd} + B_2 \left(\frac{a}{cd}\right)^2 + B_3 \left(\frac{a}{cd}\right)^3 + \cdots \right), $$

where $B_k, k \geq 1$, is an effectively computable polynomial in $x = d/a$ of degree $2k-1$. Furthermore,

$$ \begin{aligned} B_1 &= \frac{2}{3}(x+1), \\ B_2 &= -\frac{4}{135}(x+1)(x-2)(x-\tfrac{1}{2}), \\ B_3 &= \frac{8}{2835}(x+1)(x-2)(x-\tfrac{1}{2})(x^2-x+1), \\ B_4 &= \frac{16}{8505}(x+1)(x-2)(x-\tfrac{1}{2})(x^2-x+1)^2. \end{aligned} $$

Evans [31] has given an excellent discussion of several of Ramanujan's asymptotic expansions, in particular, with emphasis on hypergeometric functions.

For a survey of hypergeometric and basic hypergeometric series emphasizing Ramanujan's contributions, see Askey's paper [5].

8. q–Series

As customary, set

$$ (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \qquad n \geq 0, $$

and

$$ (a; q)_\infty = \lim_{n \to \infty} (a; q)_n, \qquad |q| < 1. $$

Ramanujan discovered the Rogers–Ramanujan identities in India, and although he did not prove them until sometime after he reached England, he had derived four pieces of evidence, found in the publication of the lost notebook [56, p. 358], for the validity of the first identity,

$$ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}. \tag{8.1} $$

In particular, as $q \to 1-$, both sides of (8.1) are asymptotically equal to

$$\exp\left(\frac{\pi^2}{15(1-q)}\right).$$

In both the third and lost notebooks, Ramanujan offers an asymptotic formula for a general class of q–series including that in (8.1).

Theorem. *Let $a > 0, |q| < 1$, and b and c be integers with $b > 0$. Let z denote the positive root of $az^{2b} + z = 1$. Then, as $q \to 1-$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q;q)_n} \sim \exp\left(-\frac{1}{\log q}\left(\mathrm{Li}_2(az^{2b}) + b\log^2 z\right)\right.$$
$$\left. + c\log z - \frac{1}{2}\log\left(z + 2b(1-z)\right)\right),$$

where $\mathrm{Li}_2(s)$ denotes the dilogarithm, defined by (4.2).

Two proofs are given in [12, pp. 269–284], with the better one due to R. McIntosh.

With the exception of the Rogers–Ramanujan identities, Ramanujan's ${}_1\psi_1$ summation, found as Entry 17 in Chapter 16 [11, p. 32], is undoubtedly his most famous result in the theory of q–series.

Further results on q–series may be found in [11, Chapter 16] and [12, Chapter 27].

9. Continued Fractions

The Rogers–Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots = q^{1/5}\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \qquad |q| < 1, \tag{9.1}$$

is the only continued fraction that Ramanujan examined in print [53], [54, pp. 214–216]. However, his notebooks contain almost 200 contributions to continued fractions, and it seems to us that no person in the history of mathematics possessed the skills that Ramanujan had in determining continued fractions for various functions or finding closed form representations for continued fractions.

In his first and second letters to Hardy, Ramanujan communicated the values of $R(e^{-2\pi}), R(-e^{-\pi})$, and $R(e^{-2\pi/\sqrt{5}})$ [54, pp. xxvii, xxviii]. Other values can be found in his first and lost notebooks. For example, on page 311 of his first notebook, Ramanujan claims that

$$R(e^{-8\pi}) = \sqrt{c^2+1} - c, \tag{9.2}$$

where

$$2c = 1 + \frac{a+b}{a-b}\sqrt{5},$$

$a = 3 + \sqrt{2} - \sqrt{5}$, and $b = (20)^{1/4}$. The first proof of (9.2) was given by H. H. Chan [16] and the author. In his first letter to Hardy, Ramanujan also asserted that

$R(e^{-\pi\sqrt{n}})$ "can be exactly found if n be any positive rational quantity." Watson [68] vaguely discussed this claim and claimed that $R(e^{-\pi\sqrt{n}})$ is an algebraic number. However, Chan, L.–C. Zhang, and the author [19] have proved that $R(e^{-\pi\sqrt{n}})$ is always a unit. K. G. Ramanathan [47], [48], [49], [50] has also extensively studied the values of $R(q)$. In his third notebook, Ramanujan also examines $R(q)$ when $|q| > 1$ and when q is a primitive nth root of unity. For $|q| > 1$, in contemporary language, Ramanujan asserted that, amazingly, the odd approximants of $R(q)$ approach $q^{1/5}/R(-1/q)$, and the even approximants approach $R(q^4)/q^{4/5}$, proved for the first time in [3]. If $q = \exp(2\pi im/n)$, where $(m, n) = 1$, Ramanujan asserted that $R(q)$ diverges if n is a multiple of 5 and converges otherwise. Moreover, when convergent, Ramanujan (somewhat imprecisely) evaluated $R(q)$. Unaware of Ramanujan's claims, I. Schur [61] first proved these assertions in 1917.

Ramanujan recorded several other continued fractions of the type (9.1) in the unorganized part of his second notebook and in his third notebook. For example,

$$\frac{1}{1} + \frac{q+q^2}{1} + \frac{q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^8}{1} + \cdots = \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}, \qquad |q| < 1. \tag{9.3}$$

The first proof of (9.3) was given in 1936 by A. Selberg [62], who derived continued fractions for a large class of quotients of q–series, and so proved several results of Ramanujan, unaware that they were in Ramanujan's still unpublished notebooks. Further references to (9.3) and other continued fractions akin to (9.1) and (9.3) can be found in [3, in particular, pp. 20–23].

Ramanujan discovered continued fractions for several products of gamma functions. We cite one of several such theorems [10, p. 155, Entry 33].

Theorem. *Let x, m, and n be complex. If either m or n is an integer, or if* Re $x > 0$, *then*

$$\begin{aligned}
&\big\{\Gamma(\tfrac{1}{2}(x+m+n+1))\Gamma(\tfrac{1}{2}(x-m-n+1)) \\
&\qquad -\Gamma(\tfrac{1}{2}(x+m-n+1))\Gamma(\tfrac{1}{2}(x-m+n+1))\big\} \div \\
&\big\{\Gamma(\tfrac{1}{2}(x+m+n+1))\Gamma(\tfrac{1}{2}(x-m-n+1)) \\
&\qquad +\Gamma(\tfrac{1}{2}(x+m-n+1))\Gamma(\tfrac{1}{2}(x-m+n+1))\big\} \\
&= \frac{mn}{x} + \frac{(m^2-1^2)(n^2-1^2)}{3x} + \frac{(m^2-2^2)(n^2-2^2)}{5x} + \cdots.
\end{aligned} \tag{9.4}$$

Many interesting continued fractions are limiting cases of these general continued fractions for products of gamma functions. For example [10, p. 145],

$$\pi = \frac{4}{1} + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \cdots$$

is Lord Brouncker's continued fraction for π, and [10, p. 155]

$$\zeta(3) = 1 + \frac{1}{2\cdot 2} + \frac{1^3}{1} + \frac{1^3}{6\cdot 2} + \frac{2^3}{1} + \frac{2^3}{10\cdot 2} + \cdots$$

is the continued fraction employed by R. Apéry [4] in his renowned proof of the irrationality of $\zeta(3)$.

The most celebrated continued fraction of the type (9.4) is Entry 40 of Chapter 12 in the second notebook [10, p. 163], which is a terminating continued fraction involving five parameters and which was first established by G. N. Watson [71]. It was not until 1991 that D. Masson [46] proved a nonterminating version of Entry 40 and a companion result as well.

Proofs of Ramanujan's continued fractions for products of gamma functions have been given by a variety of methods, but we probably still do not know Ramanujan's methods. In recent years, Masson [44], [45], [46] and L.–C. Zhang [75] have developed a unified approach through generalized hypergeometric series, and this may be closest to Ramanujan's ideas.

10. Theta Functions and Modular Equations

Ramanujan's approach to the theory of theta functions does not appear to have been influenced by any other writer. His general theta function

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1, \tag{10.1}$$

is an alternative formulation of the general classical theta function which includes as special cases $\vartheta_n(z,q), 1 \le n \le 4$ [73, pp. 463, 464], but for Ramanujan's purposes (10.1) is the most useful definition. Ramanujan's notation immediately yields the symmetry $f(a,b) = f(b,a)$. The three most important special cases of (10.1) for Ramanujan are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q) := \tfrac{1}{2} f(1,q) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} = (q;q)_\infty = q^{-1/24}\eta(z), \tag{10.2}$$

where $q = \exp(2\pi i z), \operatorname{Im} z > 0, \eta(z)$ denotes the Dedekind eta–function, and the penultimate equality in (10.2) gives Euler's famous pentagonal number theorem.

Ramanujan derived an enormous number of theta function identities, many of them classical but also many of them original. We offer three examples. For $|q| < 1$,

$$\frac{\psi^3(q)}{\psi(q^3)} = 1 + 3\sum_{n=0}^{\infty} \left(\frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+5}}{1-q^{6n+5}} \right). \tag{10.3}$$

Our proof of (10.3) in [14, pp. 18–20] is preferable to that in [11, pp. 118–119, 228–229] and uses another identity of Ramanujan,

$$\sum_{n=-\infty}^{\infty} (3n+1) q^{3n^2+2n} = (q^2;q^2)_\infty (q;q^2)_\infty^2 (q^4;q^4)_\infty^2 = \psi(q^2) f^2(-q),$$

which was perhaps first proved by B. Gordon [33]. Ramanujan derived several identities in the same spirit of (10.3), and they can be found at several places in [11]. Some are classical identities connected with formulas for the numbers of representations of a positive integer n by the quadratic forms x^2+y^2, x^2+3y^2, and x^2+xy+y^2.

Some of Ramanujan's most beautiful and useful theta function identities are his 23 identities for the Dedekind eta–function [12, pp. 204–244]. We offer one example. Let

$$P = \frac{f^2(-q)}{q^{1/6}f^2(-q^3)} \qquad \text{and} \qquad Q = \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}.$$

Then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

Applications of Ramanujan's eta–function identities can be found in [16] and [20].

At scattered places in the first and second notebooks, Ramanujan also recorded specific values for $\varphi(q)$. For example, on page 297 of his first notebook, Ramanujan wrote

$$\frac{\varphi^2(e^{-7\pi})}{\varphi^2(e^{-\pi})} = \frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{14}(28)^{1/8}. \tag{10.4}$$

It is classical that $\varphi(e^{-\pi}) = \pi^{1/4}/\Gamma(\frac{3}{4})$ [73, p. 525], [11, p. 103], and so (10.4) gives an exact evaluation of $\varphi(e^{-7\pi})$. Proofs of (10.4) and Ramanujan's other values for $\varphi(q)$ can be found in our paper with Chan [17]. Moreover, Chan, Zhang and the author have also shown that $\varphi(e^{-n\pi})/\varphi(e^{-\pi})$ is algebraic for every positive integer n [21].

Let

$$K := K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{10.5}$$

denote the complete elliptic integral of the first kind, where $k, 0 < k < 1$, is called the *modulus*. It is easy to show that

$$K(k) = \frac{\pi}{2}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k^2). \tag{10.6}$$

One of the classical and most important theorems in the theory of elliptic functions is the "inversion formula" (10.8) below. As usual in the theory of elliptic functions, set

$$q = \exp\left(-\pi\frac{K'}{K}\right) = \exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}\right), \tag{10.7}$$

where K is defined by (10.5) and $K' = K(k')$, where $k' = \sqrt{1-k^2}$ is called the *complementary modulus*. Then

$$\varphi^2(q) = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k^2) =: z. \tag{10.8}$$

For Ramanujan's proof of (10.8), see [11, pp. 98–101] or [14, pp. 24–27]. Formula (10.8) and several elementary identities for theta functions can be utilized to evaluate theta functions in terms of the parameters q, k, and z. In Entries 10–12 of Chapter 17 in his second notebook [11, pp. 122–124], Ramanujan offers a "catalogue" of these evaluations. For example,

$$\varphi(-q^2) = \sqrt{z}(1-k^2)^{1/8} \qquad \text{and} \qquad \psi(q^2) = \tfrac{1}{2}\sqrt{z}(k^2/q)^{1/4}.$$

This catalogue serves as the foundation for Ramanujan's modular equations.

There are several definitions of a modular equation. The one frequently given in the theory of modular forms [58, p. 118] is slightly different from that used by Ramanujan, and we give the latter definition now. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that, for some fixed positive integer n,

$$\frac{L'}{L} = n\frac{K'}{K}. \tag{10.9}$$

Then a modular equation of degree n is a relation between the moduli k and ℓ which is implied by (10.9). The multiplier m, which often appears in modular equations, is defined by

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{10.10}$$

In view of (10.7), if we set $q' = \exp(-\pi L'/L)$, then (10.9) is equivalent to the relation $q^n = q'$. It is well known that k and ℓ can be expressed in terms of theta functions [11, p. 102], and so a modular equation of degree n can also be viewed as an identity among theta functions at the arguments q and q^n. Ramanujan did not employ the classical notation; he set $\alpha = k^2$ and $\beta = \ell^2$. Ramanujan also derived modular equations involving four moduli. These "mixed" modular equations, or modular equations of composite degree, arise from a triple of relations of the form (10.9). For a more precise definition of "mixed" modular equation, see [11, p. 325] or [14, p. 30].

In one sense, the theory of modular equations began with the transformations of Gauss and Landen giving rise to modular equations of degree 2 [14, pp. 30, 31]. However, the history of the subject is usually considered to have begun in 1825 with Legendre's modular equation of degree 3 [41]. In the next 100 years, many modular equations were derived by E. Fielder, R. Fricke, A. G. Greenhill, C. Guetzlaff, M. Hanna, C. G. J. Jacobi, F. Klein, R. Russell, L. Schläfli, H. Schröter, H. Weber, and others. However, in his notebooks, Ramanujan perhaps recorded more modular equations than those of his predecessors combined. Chapters 19–21 in his second notebook are almost completely devoted to modular equations. Dozens of others can be found scattered among the unorganized portions of both his first and second notebooks. See [11, Chapters 19–21], [12, Chapter 25], and [13, Chapter 36] for proofs of all of Ramanujan's modular equations in his notebooks.

We next cite a few examples. Legendre's modular equation of degree 3, also discovered by Ramanujan [11, p. 230, Entry 5(ii)], is given by

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1.$$

A modular equation in $\alpha\beta$ and $(1-\alpha)(1-\beta)$ is often called an *irrational modular equation*. Such modular equations are often the simplest and very useful, and they have been derived for several values of n, but as n increases, they become exceedingly more complicated.

Another useful type of modular equation is the *Schläfli modular equation*. Let

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \qquad \text{and} \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}.$$

Then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0 \tag{10.11}$$

is a modular equation of degree 3, first established by L. Schläfli [60] in 1870 and rediscovered by Ramanujan [11, p. 231, Entry 5(xii)]. Such equations are very important in the calculation of class invariants, discussed in Section 12.

Our third example is a modular equation of degree 17 [11, p. 397, Entry 12(iii)]:

$$\begin{aligned} m = {} & \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \\ & - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left(1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right). \end{aligned} \tag{10.12}$$

Many methods have been developed to prove modular equations, but a method that may be applicable for one type of equation or for one degree may be inapplicable for another type of equation or degree. Generally, a modular equation is expressed, via the aforementioned catalogue of "evaluations," as a theta function identity, and the task is then to prove the theta function identity. We do not know Ramanujan's methods, but he apparently used basic elementary properties of theta functions, the Jacobi triple product identity, the quintuple product identity, transformation formulas for theta functions, his ${}_1\psi_1$ summation formula, and Lambert series. Almost certainly, he used Schröter's formulas for theta functions [11, pp. 65–76]. He also may have used parametrizations of $\alpha, \beta, 1-\alpha$, and $1-\beta$ in conjunction with previously derived modular equations to find new modular equations. However, for many of Ramanujan's modular equations, (10.12), for example, we have been unable to use tools that might have been known to Ramanujan and so have had to resort to the theory of modular forms. Although this is a very powerful tool, it is pedagogically unsatisfactory, because the modular equation needs to be known in advance to effect a proof.

Andrews [2] and L.–C. Shen [63], [64], [65], [66] have recently found new proofs of some of Ramanujan's modular equations of degrees 3 and 5.

Ramanathan's paper [51] and our lecture notes [14] provide introductions to much of Ramanujan's work on modular equations.

11. Ramanujan's Theories of Elliptic Functions to Alternative Bases

In his famous paper on modular equations and approximations to π [52], [54, pp. 23–39], Ramanujan offers several series representations for $1/\pi$, for example (see

also [26]),

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(6n+1)(\frac{1}{2})_n^3}{(n!)^3 4^n},$$

where $(a)_n := a(a+1)\cdots(a+n-1), n \geq 0$. He asserts [54, p. 37] that some of his formulas are derived from " ... corresponding theories in which q is replaced by one or other of the functions ... ." In the notation (10.6) and (10.7), Ramanujan replaced q by

$$q_r := \exp\left(-\pi \csc(\pi/r) \frac{{}_2F_1(\frac{1}{r}, \frac{1-r}{r}; 1; 1-k^2)}{{}_2F_1(\frac{1}{r}, \frac{1-r}{r}; 1; k^2)}\right), \tag{11.1}$$

where $r = 3, 4$, or 6. Note that $q_2 = q$, as given by (10.7). The first six pages in the unorganized part of the second notebook [55, pp. 257–262] are devoted to these theories. Except for a few entries verified by K. Venkatachaliengar [67], no one had examined this material until S. Bhargava, F. G. Garvan, and the author [15] completed proofs for all these entries. The theories for $r = 4$ and 6 hinge chiefly on the classical theory when $r = 2$. The theory for $r = 3$ is more interesting and difficult. Fortunately, shortly after the authors began their efforts, J. M. and P. B. Borwein [27] (see also [28]) discovered a cubic analogue of a famous identity of Jacobi [11, p. 40, Entry 25(vii)] for their cubic theta functions. This was necessary for us to obtain an analogue of (10.8). We now state the Borweins' cubic theta functions, their cubic theta function identity, and the analogue of (10.8).

Let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2},$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2},$$

where $\omega = \exp(2\pi i/3)$. Then [27], [28]

$$a^3(q) = b^3(q) + c^3(q). \tag{11.2}$$

The analogue of (10.8) is given by [15, Theorem 2.10]

$$a(q_3) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; k^2\right) =: z_3. \tag{11.3}$$

Besides (11.2), a key ingredient in proving (11.3) is the cubic transformation formula

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x^3\right), \tag{11.4}$$

where $0 \le x < 1$. This transformation is found in Ramanujan's notebooks and was rediscovered by the Borweins [27].

The identity (11.2) is not found in the notebooks, but a short proof of it can be constructed by using material from the notebooks [15, Lemma 2.1, Theorem 2.2], [14, Theorem 13.3]. It appears inconceivable to us that Ramanujan could have derived (11.3) without knowing (11.2) or an equivalent result.

Ramanujan, in fact, offered several new transformation formulas for hypergeometric functions on the aforementioned six pages. Moreover, we have used computer algebra to derive generalizations of these transformations [15]. See Garvan's paper [32] to learn how MAPLE was utilized in these discoveries.

The problem of inverting elliptic integrals is central in the theory of elliptic functions. The following inversion formula of Ramanujan is an analogue of one of these inversion formulas.

Theorem. *Let q_3 and z_3 be given by (11.1) and (11.3), respectively. For $0 \le \varphi \le \pi/2$ and $0 \le x < 1$, define $\theta = \theta(\varphi)$ by*

$$\theta z_3 = \int_0^{\varphi} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x\sin^2 t\right) dt. \tag{11.5}$$

Then, for $0 \le \theta \le \pi/2$,

$$\varphi = \theta + 3\sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n(1+2\cosh(ny))},$$

where $q_3 =: \exp(-y)$.

Recalling that [10, p. 99, Entry 35(iii)]

$${}_2F_1\left(\frac{1}{2}+a, \frac{1}{2}-a; \frac{1}{2}; x^2\right) = (1-x^2)^{-1/2}\cos(2a\arcsin x),$$

where a is any complex number, we see that, if $a = 0$ and x^2 is replaced by $x\sin^2 t$, then (11.5) is a natural analogue of the inversion formula for the incomplete elliptic integral of the first kind.

Ramanujan recorded several modular equations in the alternative theories. For example, if $r = 3$, then [15, Theorem 7.5]

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1$$

is a modular equation of degree 3.

12. Class Invariants

Let $\chi(q) = (-q; q^2)_\infty$. If $q = \exp(-\pi\sqrt{n})$, where n is any positive rational number, then Ramanujan's two class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q).$$

In the notation of Weber [72], $G_n =: 2^{-1/4}\mathfrak{f}(\sqrt{-n})$ and $g_n =: 2^{-1/4}\mathfrak{f}_1(\sqrt{-n})$. It is well known that G_n and g_n are algebraic; for example, see Weber's book [72] or Cox's book [30, p. 214, Theorem 10.23; p. 257, Theorem 12.17].

At scattered places in his first notebook, Ramanujan recorded the values for 107 class invariants. On pages 294–299 in his second notebook, Ramanujan gave a table of values for 77 class invariants, three of which are not found in the first notebook. After coming to England, Ramanujan became aware of Weber's work, and so his table of 46 class invariants in [52] does not contain any that are found in Weber's book [72]. Watson wrote eight papers on class invariants, and, in particular, in [69] and [70], he established 24 of Ramanujan's class invariants. After Watson's work, 18 of Ramanujan's class invariants remained to be verified, but these have now been proved in two papers [18], [20] with H. H. Chan and L.–C. Zhang. We briefly describe this work.

In [18], five of Ramanujan's invariants are proved by appealing to two new theorems that relate G_{9n} with G_n, and g_{9n} with g_n, respectively. The modular equation (10.11) is the primary result used in the proofs. Thus, for example, using the known value for G_{17}, we showed that

$$G_{153} = \left(\sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}\right)^2 \left(\sqrt{\frac{37+9\sqrt{17}}{4}} + \sqrt{\frac{33+9\sqrt{17}}{4}}\right)^{1/3}.$$

The remaining thirteen values of G_n were more difficult to prove. In each case, the class number of $\mathbb{Q}(\sqrt{-n})$ equals 8, and there are two classes per genus. Three methods were employed. The first utilizes Ramanujan's modular equations, and so is in the spirit of Ramanujan. For example, to prove that

$$G_{65} = \left(\frac{\sqrt{13}+3}{2}\right)^{1/4} \left(\frac{\sqrt{5}+1}{2}\right)^{1/4} \left(\sqrt{\frac{9+\sqrt{65}}{8}} + \sqrt{\frac{1+\sqrt{65}}{8}}\right)^{1/2},$$

a modular equation of degree 5, similar to (10.11), and one of Ramanujan's eta–function identities are employed. However, we were able to use modular equations to prove only six of the thirteen invariants. The second method depends upon Kronecker's limit formula and can be used to establish all thirteen class invariants, but these ideas were not known to Ramanujan. The third employs class field theory to make Watson's "empirical process" in [69] rigorous. Although Watson was confident that Ramanujan employed this "empirical process," we disagree. In summary, Ramanujan's proofs for several of his class invariants remain elusive.

Many mathematicians have either individually or in collaboration with the author proved many of Ramanujan's difficult claims in the notebooks. The following list is incomplete, and we apologize to those whose names we have omitted. Each has proved theorems of Ramanujan that would remain unproved without their efforts. We offer our sincere thanks to: George E. Andrews, Richard A. Askey, Gennady Bachman, S. Bhargava, Anthony J. Biagioli, David Bradley, Heng Huat Chan, Henri Cohen, Ronald J. Evans, Frank G. Garvan, James L. Hafner, Lisa Lorentzen, Kenneth S. Williams, Don Zagier, and Liang–Cheng Zhang.

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Ramanujan's Notebooks

The history of the notebooks, in brief, is the following:

Ramanujan had noted down the results of his researches, without proofs, (as in *A Synopsis of Elementary Results, a book on pure Mathematics,* by G.S. Carr), in three notebooks, between the years 1903 - 1914, before he left for England. These were the notebooks which he showed to his benefactors to convince them about his abilities as a mathematician. The results in these notebooks were organized by him. The first notebook has 16 chapters in 134 pages. The second is a revised, enlarged version of the first, containing 21 chapters in 252 pages. The third notebook contains 33 pages of unorganized material (included at the end of volume 2 of the facsimile edition [VII]). Ramanujan took these notebooks with him to Cambridge. But, in one of his letters to a friend, he wrote that he had no time to look into them and most probably he did not put them to use during his five year stay abroad.

The Preface to the *Collected Papers of Srinivasa Ramanujan* ([III], p.ix) contains the following appraisal of the work in the notebooks:

> *There is still a large mass of unpublished material. None of the contents of Ramanujan's notebooks has been printed, unless incorporated in later papers, except that one chapter, on generalized hypergeometric series, was analyzed by Hardy* [37] *in the Proceedings of the Cambridge Philosophical Society. This chapter is sufficient to show that, while the notebooks are naturally unequal in quality, they contain much which should certainly be published. It would be a very formidable task to work through them systematically, select particular passages, and edit these with adequate comment, and it is impossible to print the notebooks as they stand without further monetary assistance. The singular quality of Ramanujan's work and the romance which surrounds his career, encourage us to hope that this volume may enjoy sufficient success to make possible the publication of another.*

Hardy tried to persuade the University of Madras to undertake such a task and in turn, Prof. G.N. Watson was requested by the Madras University, in 1931, to edit the notebooks in a suitable form for publication. This was a formidable task since the notebooks contained about 4000 theorems.

Prof. Watson gave a lecture [39] on Ramanujan's notebooks at the London Mathematical Society in 1931, and published several papers between 1928–36, motivated by Ramanujan's work. Watson's Presidential address to the London Mathematical Society, in 1935, was on *mock-theta functions*, discovered by Ramanujan during the period when he was terminally ill (March 1919 – April 1920) after his return to India. Watson undertook the task of editing the notebooks with Prof. B.M. Wilson and

> *they focussed their attention on the second notebook. Chapters 2–13 were to be edited by Wilson, and Watson was to examine chapters 14–21. Unfortunately, Wilson passed away prematurely in 1935 at the age of 38.* (Berndt [40]).

This possibly put an end to the joint effort of Watson and Wilson to edit the notebooks.

In 1957, with monetary assistance from Sir Dadabai Naoroji Trust, at the instance of Prof. Homi J. Bhabha and Prof. K. Chandrasekaran of the Tata Institute of Fundamental Research, the Tata Institute published a facsimile edition of the notebooks of Ramanujan in two volumes. The reproduction is thoroughly faithful to the original. About a thousand copies were made then and in the centenary year of Ramanujan's birth, 1987, Springer - Narosa reprinted the same.

The formidable task of truly editing the notebooks –*viz.* to either prove each of the results or provide references to the literature where the proofs may be found – which was started by G. N. Watson and B. M. Wilson but never completed, was taken up in right earnest by Prof. Bruce C. Berndt, who has been publishing a series of papers, since 1981, by himself or in collaboration with others, including Indian Mathematicians Padmini T. Joshi, C. Adiga and S. Bhargava. He has also utilized the earlier work of G.N. Watson and B.M. Wilson on the notebooks with the Trinity College. This dedicated work of Berndt, published by Springer-Verlag [VIII], is now available in five parts, the first of which appeared in 1985 and the fifth, in 1997.

At the beginning of his work, in Part I, Berndt [34] points out:

> *The notebooks were originally intended primarily for Ramanujan's own personal use and not for publication ... Some of Ramanujan's incorrect 'theorems' in number theory found in his letters to Hardy have been well publicized. Thus, perhaps, some think that Ramanujan was prone to making errors. However, such thinking is erroneous. The notebooks contain scattered errors. Especially if one takes into account the roughly hewn nature of the material and his frequently formal arguments, Ramanujan's accuracy is amazing.*

Resurgence of interest

After Ramanujan returned to India, though he was seriously ill and bedridden for most of the time, he was working continuously. His life-long habit had been to work on a slate and transfer the final results to the notebook or sheets of paper. His brother-in-law has stated [59] that *a vast quantity of papers containing his notes was handed over to the University after his death*. Three days after Ramanujan died, his brother S. Lakshminarasimhan, in a letter to Prof. Hardy, dated 29th April 1920, informing him about his demise, wrote :

> *All his Mss that were in his trunk were handed over on the day of his death to Mr. Ramachandra Row's son in law, since the former is at Nilgiris. Not only those but also the journals, magazines - all he possessed except the books which he had, were taken away by them.*

In a letter to Prof. Hardy, dated 3 Dec. 1920, Mr. R. Ramachandra Row, wrote:

> *Ramanujan's M.S.S whatever they are are with me and will be handed over intact to the University of Madras who I understand is already in correspondence with you regarding the methods of publication.*

These papers reached Prof. Hardy, who should have passed them on to Prof. G.N. Watson and hence to their final destination – viz. the estate of Prof. Watson, in the underground cellar of Trinity College – after he wrote several papers inspired Ramanujan's results, until their discovery in 1976 by George E. Andrews.

Prof. Watson died in Feb. 1965. A decade later, in the spring of 1976, when Prof. George Andrews [53] of the Pennsylvania State University was going through the estate of Watson, he discovered a box of papers.

> *The most interesting item in this box was a manuscript of more than one hundred pages in Ramanujan's distinctive handwriting which contains over six hundred mathematical formulae listed one after the other without proof. It is my contention that this manuscript, or notebook, was written during the last year of Ramanujan's life after his return to India from England. My evidence for this assertion is all indirect; in the words of Stephen Leacock, 'It is what we call circumstantial evidence — the same that people are hanged for'* .

These are now referred to as the contents of the *Lost* notebook of Ramanujan. Andrews assessed ([101], p.xii):

> *While it is impossible to categorize the various formulas completely, a rough approximation of its contents is the following:*
>
> *q-series and related topics including mock θ-functions: 60%*
> *Modular equations and relation, singular moduli: 30%*
> *Integrals, Dirichlet series, congruences, asymptotics, misc.: 10%*
>
> *I give only this rough break down because of the chaotic nature of this manuscript. On many pages there are fragments of numerical computations and infinite series running off in all directions. It may be possible eventually to make sense out of some of these pages; if so, the above percentage breakdown may change. . . .*
> *In any event it is clear that q-series investigations make up the bulk of the work in in the Lost Notebook. Indeed I count about 380 formulae that I consider to belong more to q-series than to modular relations or other topics.*

Sixty-seven years after the death of Ramanujan, due to the discovery of the 'lost' notebook by Prof. George Andrews, in 1976, and the editing of the three Notebooks of Ramanujan by Prof. Bruce Berndt, there has been a resurgence of interest in the work of Ramanujan. These notebooks of Ramanujan have formed the basis for numerous papers by many mathematicians, who gave proofs of the theorems and conjectures of Ramanujan obtained by him through his intuition and sheer brilliance. This is a singular and unparalleled phenomenon in the annals of mathematics. In the centenary year of his birth, 1987, several conferences were held in many countries to focus attention on his work and the recent developments. Mystery still surrounds *some of his work, mostly in the theory of elliptic functions*, wherein

> *it is not possible, after all the work of Watson and Mordell, to draw the line between what he may have picked up somehow and what he must have found for himself.* ([IV], p.10).

Hardy [60] regretted that he could easily have but did not find out all the details from Ramanujan himself since he

> *was quite able and willing to give a straight answer to a straight question, and not in the least disposed to make a mystery of his achievements*.

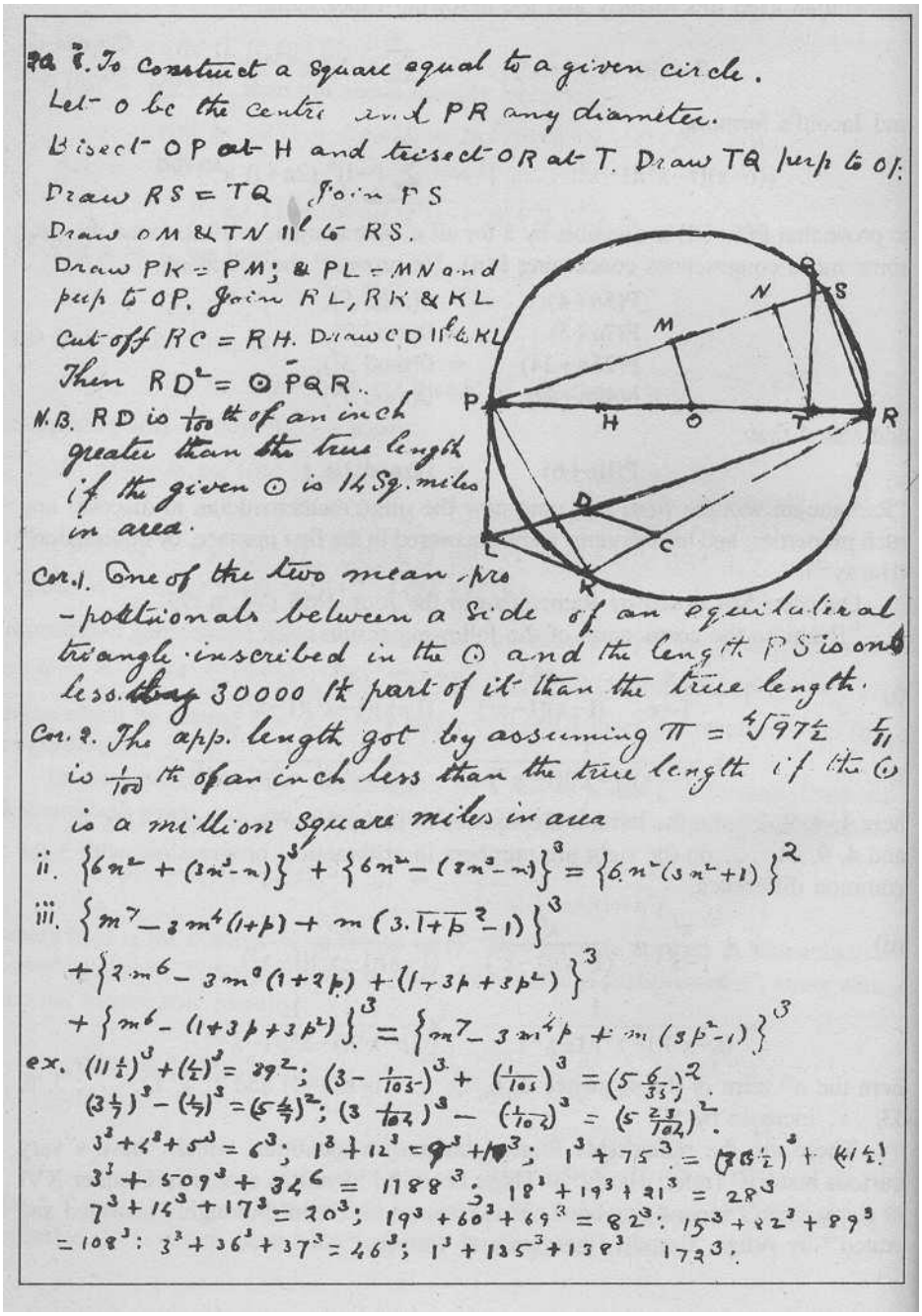
However, Hardy [60] argues that Ramanujan

> *was not particularly interested in his own history or psychology; he was a mathematician anxious to get on with the job. And after all I too was a mathematician, and a mathematician meeting Ramanujan had more interesting things to think about than historical research. It seemed ridiculous to worry him about how he had found this or that known theorem, when he was showing me half a dozen new ones almost every day.*

A Page from the second Notebook

To provide a feeling for Ramanujan's notebooks, we reproduce here a page [61] from his second Notebook, Chapter XVIII. There is no particular reason for choosing this page except that it contains an entry which has become a 'folklore' which illustrates the attachment Ramanujan had to numbers prompting Littlewood to state that to Ramanujan every number is a *personal friend*.

There are 2 parts shown in this page. The first part is concerning the geometrical construction of a square whose area is equal to that of a given circle. This is a rare piece of Ramanujan's work in geometry, since he did very little work in geometry and perhaps did not have much interest in it. Several attempts on this problem have been made from early times by (ancient Greek and Hindu) mathematicians and it has been shown that it is impossible



To Construct a square equal to a given circle.

Let O be the centre and PR any diameter.

Bisect OP at H and trisect OR at T. Draw TQ perp to OP.

Draw RS = TQ. Join PS

Draw OM & TN ll^{l} to RS.

Draw PK = PM; & PL = MN and perp to OP. Join RL, RK & KL

Cut off RC = RH. Draw CD ll^{l} to KL

Then RD^{2} = ⊙ PQR.

N.B. RD is $\frac{1}{100}$ th of an inch greater than the true length if the given ⊙ is 14 Sq. miles in area.

Cor. 1. One of the two mean proportionals between a side of an equilateral triangle inscribed in the ⊙ and the length PS is one less. ~~thing~~ 30000 th part of it than the true length.

Cor. 2. The app. length got by assuming $\pi = \sqrt[4]{97\frac{1}{2}}$ $\frac{5}{11}$ is $\frac{1}{100}$ th of an inch less than the true length if the ⊙ is a million square miles in area

ii. $\{6n^2 + (3n^2-n)\}^3 + \{6n^2 - (3n^2-n)\}^3 = \{6n^2(3n^2+1)\}^2$

iii. $\{m^7 - 3m^4(1+p) + m(3\sqrt{1+p^2}-1)\}^3$

$+\{2m^6 - 3m^3(1+2p) + (1+3p+3p^2)\}^3$

$+\{m^6 - (1+3p+3p^2)\}^3 = \{m^7 - 3m^4p + m(3p^2-1)\}^3$

ex. $(11\frac{1}{2})^3 + (\frac{1}{2})^3 = 89^2$; $(3 - \frac{1}{103})^3 + (\frac{1}{103})^3 = (5\frac{6}{357})^2$.

$(3\frac{1}{7})^3 - (\frac{1}{7})^3 = (5\frac{4}{7})^2$; $(3\frac{1}{102})^3 - (\frac{1}{102})^3 = (5\frac{23}{102})^2$.

$3^3 + 4^3 + 5^3 = 6^3$; $1^3 + 12^3 = 9^3 + 10^3$; $1^3 + 75^3 = (38\frac{1}{2})^3 + 41\frac{1}{2}^3$

$3^3 + 509^3 + 346^3 = 1188^3$; $18^3 + 19^3 + 21^3 = 28^3$

$7^3 + 14^3 + 17^3 = 20^3$; $19^3 + 60^3 + 69^3 = 82^3$; $15^3 + 82^3 + 89^3 = 108^3$; $3^3 + 36^3 + 37^3 = 46^3$; $1^3 + 135^3 + 138^3 = 172^3$.

Fig. caption: The last three lines of p. 224 and the p. 225 from the second notebook of Ramanujan, chapter XVIII. ([61]).

to *square the circle*, in the Euclidian sense[1]. Hardy has commented that he found in Ramanujan's collection of books in Cambridge some books by 'quacks' on this theme.

Ramanujan published a paper [62] entitled, "*Squaring the circle*" in the Jour. of the I.M.S. in 1913, in which he gives a geometrical construction for finding the length of the side of a square whose area equals that of the circle. He also reproduced this and another geometrical construction for π in his later paper [63] on "*Modular equations and approximations to* π". In this paper, he deduced several formulae[2] for π like:

$$\frac{63}{25}\frac{17+15\sqrt{5}}{7+15\sqrt{5}} = 3.14159265380...$$

$$\pi = \sqrt{\frac{24}{142}}\log\left(\sqrt{\frac{10+11\sqrt{2}}{4}} + \sqrt{\frac{10+7\sqrt{2}}{4}}\right)$$

$$\pi = 12\sqrt{190}\,\log\{(2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\}$$

formulae correct to 9, 16 and 18 decimal places; and

$$\frac{1}{\pi} = 2\sqrt{2}\left[\frac{1103}{99^2} + \frac{27493}{99^6}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \frac{53883}{99^10}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4\cdot 8^2} + \cdots\right]^{-1/4}$$

$$= \frac{2\sqrt{2}}{99^2}\sum_{n=0}^{\infty}\frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n(1)_n}(1103+26390n)\left(\frac{1}{99}\right)^{4n},$$

which he asserted would be "*rapidly convergent*", for the first term itself gives the sum to 8 decimal places. In 1986, Borwein and Borwein [64], two computer scientists used a version of Ramanujan's formula to calculate π to

[1] It is the ancient Greeks who set-up the circle-squaring problem with the following two conditions [83]: the solution should use only a straightedge and compass, so that the proofs can be reduced to Euclid's theorems and the solution must use only a finite number of steps. If, in fact, $\pi r^2 = a^2$, where r is the radius of the circle and a is the length of the side of a square, then it follows that $\pi = (a/r)^2$, or that π is rational. In fact, π has been shown to be not only an *irrational number* but also, proved by Ferdinand von Lindermann, in 1882, to be a *transcendental* number. Furthermore, the phrase 'squaring the circle' in common usage suggests a project doomed to failure!

[2] Note that Ramanujan has used the equality sign for his approximations to π even though he states explicitly that the formulae are correct only to a certain number of decimal places.

17 million places and found that the formula converges on the exact value with far greater efficiency than any previous method[3]. This success proved that Ramanujan's insight was correct.

The next entries on this page in Ch. XVIII of his second note book are solutions of Diophantine equations:

$$X^3 + Y^3 = U^2$$

and $$X^3 + Y^3 + Z^3 = U^3$$ (Euler's equation).

which are followed by numerical examples for these equations – 6 solutions for the first and 12 solutions for the Euler's equation and one which looks like but is not a solution of the Euler's equation. Characteristically, Ramanujan does not give the values of the parameters for the examples (ref. [VIII], part III, p.199 for the examples with the values of the parameters appended).

Ramanujan's solution of Euler's equation in rational numbers given here is:

$$\begin{aligned} X &= m^7 - 3m^4(1+p) + m(2 + 6p + 3p^2), \\ Y &= 2m^6 - 3m^3(1 + 2p) + 1 + 3p + 3p^2, \\ Z &= m^6 - 1 - 3p - 3p^2, \\ U &= m^7 - 3m^4 p + m(3p^2 - 1), \end{aligned}$$

where m and p denote arbitrary numbers. He has also given another solution of this equation elsewhere (see Question 441 which follows) but neither of these two parameter solutions is the general solution[4] which contains three parameters, prompting Hardy [65] to comment that

> *Diophantine equations should have suited him, but he did comparatively little with them, and what he did was not his best.*

The second of the 12 examples given for the Euler equation on this page:

$$1^3 + 12^3 = 9^3 + 10^3 \quad (= 1729)$$

---

[3] David and Gregory Chudnovsky are two mathematicians who hold several world records for calculating the highest number of digits of π: first up to 450 million digits, then 1 billion and then 2 billion digits. The current record is over 51 billion digits (cf. For a brief history of π, [83]).

[4] The last of the examples $1^3 + 6^3 + 8^3 = 9^3$ does not belong to this class of solutions, since no values of m and p yield this solution to the Euler's equation. So, its presence reveals that perhaps, Ramanujan was aware that his solution was not the most general for Euler's equation.

corresponds to ($m = 2$, $p = 3$). This has been made famous by the following anecdote associated with it, due to Hardy [66]:

> *He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Littlewood who said that every positive integer was one of Ramanujan's personal friends. I remember going to see him once when he was lying ill in Putney. I had ridden in taxi-cab number 1729, and remarked that the number seemed to me rather a dull one, and that I hoped that it was not an unfortunate omen. 'No', he replied, 'it is a very interesting number, it is the smallest number expressible as a sum of two cubes in two different ways'. I asked him, naturally, whether he could tell me the solution of the corresponding problem for fourth powers; and he replied, after a moment's thought, that he knew no obvious example, and supposed that the first such number must be very large.*

The simplest known solution of:

$$x^4 + y^4 = z^4 + t^4$$

is Euler's: $59^4 + 158^4 = 133^4 + 134^4 = 635318657$.
Euler gave a solution involving two parameters, but no 'general' solution is known.

The anecdote reveals Ramanujan's remarkable feeling for numbers and his sharp memory which made him recall one entry out of several thousands he had made in his notebook just like that and the fact that he had not recorded in his notebook the observation he made about 1729 which came from him only when Prof. Hardy made an innocuous statement regarding the taxi number.

Though the number 1729 itself finds no explicit mention in the notebooks, the Euler's equation and/or its solution: $1^3 + 12^3 = 9^3 + 10^3$ finds mention[5] in three Questions to the Journal of the I.M.S. ([III], p.331) and in two places in his notebooks ([VII], Vol. 2, p.266 and 387). These are indicated below:

[5] I am thankful to Mr. P.K. Srinivasan for drawing my attention to these and his publications [81].

• Question 441 (V, 39)[6]:
Shew that

$$(6a^2 - 4ab + 4b^2)^3 = (3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3,$$

and find other quadratic expressions satisfying similar relations.
[*Solution* by S. Narayanan, VI, 226.]

• Question 661 (VII, 119):
Solve in integers $x^3 + y^3 + z^3 = u^6$,
and deduce the following:

$$6^3 - 5^3 - 3^3 = 2^6, \qquad 8^3 + 6^3 + 1^3 = 3^6,$$
$$12^3 - 10^3 + 1^3 = 3^6, \qquad 46^3 - 37^3 - 3^3 = 6^6,$$
$$174^3 + 133^3 - 45^3 = 14^6, \qquad 1188^3 - 509^3 - 3^3 = 34^6.$$

[*Solutions by* N.B. Mitra, XIII, 15 – 17. Additional solution and remarks by N.B. Mitra, XIV, 73 – 77.]

• Question 681 (VII, 160):
Solve in integers $x^3 + y^3 + z^3 = 1$,
and deduce the following:

$$6^3 + 8^3 = 9^3 - 1, \qquad 9^3 + 10^3 = 12^3 + 1,$$
$$135^3 + 138^3 = 172^3 - 1, \qquad 791^3 + 812^3 = 1010^3 - 1,$$
$$11161^3 + 11468^3 = 14258^3 + 1, \qquad 65601^3 + 67402^3 = 83802^3 + 1.$$

[*Partial solution by* N.B. Mitra, XIII, 17. See also N.B. Mitra, XIV, 73–77 (76–77).]

• If p, q, r are quantities so taken that $p + 3a^2 = q + 3ab = r + 3b^2 = (a + b)^2$ and m and n are any two quantities, then

$$n(mp + nq)^3 + m(mq + nr)^3 = m(np + mq)^3 + n(nq + mr)^3.$$

A particular case of the above theorem is:

$$(3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3 = (6a^2 - 4ab + 4b^2)^3.$$

([VIII], Vol.2, p.266)

[6] J. of the IMS, vol. V, p.39.

- If $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$, then

$$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3.$$

([VIII], Vol.2, p.387)

Note that in the above 4-parameter solution for the Euler's equation, it has been observed (cf. [95] (c), p.43) that for $\alpha = 3, \beta = 0, \lambda = 3$ and $\gamma = 1$, we get $12^3 + 1^3 = 10^3 + 9^3 = 1729$. The fact that Ramanujan could come out spontaneously with the statement that 1729 in the smallest number which could be expressed as the sum of two cubes in two different ways, something which he had not jotted down in his notebooks, proves that every integer is a 'personal friend' of Ramanujan as remarked by Prof. Littlewood.

Hardy's statement that two thirds of Srinivasan Ramanujan's work in India, (contained in the notebooks), consisted of rediscoveries, is refuted by Prof. Bruce C. Berndt, who states after his extensive studies that Hardy's estimate is too high. Berndt's main goal was: "*To prove each of Ramanujan's theorems*" and for known results refer to literature where proofs may be found. Tables of the contents of "Ramanujan's Notebooks", Parts I to V, edited by Prof. Berndt are given here in the following pages. The titles of the chapters gives the reader an idea about the nature and range of problems and the number of results Ramanujan had recorded in each chapter.

In the Introduction to Part V, Berndt states:

> *This volume, however, should not be regarded as the closing chapter on Ramanujan's notebooks. Instead, it is just the first milestone on our journey to understanding Ramanujan's ideas. ... It is our fervent wish that these volumes will serve as springboards for further investigations by mathematicians intrigued by Ramanujan's remarkable ideas. ... Ramanujan remarked that several of his series arose from alternative theories of elliptic functions ... The first of the three alternative theories is the most interesting and the most important, and we feel that a large body of work remains to be discovered here.*

Ramanujan's Note Books, Part I, Bruce C. Berndt ([VIII], 1985):

Ch.	Subject	# of Results
1.	Magic Squares	43
2.	Sums Related to the Harmonic series or the Inverse Tangent Function	68
3.	Combinatorial Analysis and Series Inversions	86
4.	Iterates of the Exponential Function and an Ingenius Formal Technique	50
5.	Eulerian Polynomials and Numbers, Bernoulli Numbers and the Riemann Zeta-Function	94
6.	Ramanujan's Theory of Divergent series	61
7.	Sums of Powers, Bernoulli Numbers and the Γ Function	110
8.	Analogues of the Gamma Function	108
9.	Infinite Series Identities, Transformations, and Evaluations	139
	Ramanujan's Quarterly Reports	

Ramanujan's Note Books, Part II, Bruce C. Berndt ([VIII], 1989):

Ch.	Subject	# of Results
10.	Hypergeometric Series I	116
11.	Hypergeometric Series II	103
12.	Continued Fractions	113
13.	Integrals and Asymptotic Expansions	92
14.	Infinite Series	87
15.	Asymptotic Expansions Modular Forms	94

Ramanujan's work on hypergeometric series, contained in chapters X and XI of his second notebook was edited by Hardy, as stated earlier. In chapter 4 of this book, we take a closer look at some of the beautiful work of Ramanujan on hypergeometric series.

Ramanujan's Notebooks, Part III, Bruce C. Berndt ([VIII], 1991):

Ch.	Subject	# of Results
16.	q-Series and Theta Functions	134
17.	Fundamental Properties of Elliptic Functions	162
18.	The Jacobian Elliptic Functions	135
19.	Modular Equations of Degrees 3, 5 and 7 and Associated Theta Function Identities	185
20.	Modular Equations of Higher & Composite Degrees	173
21.	Eisenstein series	45

The 100 Pages of Unorganized material after Chapter 21 of Srinivasa Ramanujan's second Notebook and the 33 Pages of his third Notebook were organized into chapters 22 to 31 by Berndt.

Ramanujan's Notebooks, Part IV, Bruce C. Berndt ([VIII], 1994):

Ch.	Subject	# of Results
22	Elementary results	47
23.	Number Theory	108
24.	Theory of Prime numbers	24
25.	Theta Function and Modular Equations	86
26.	Inversion Formulas for Lemniscate & other functions.	10
27.	Q series	9
28.	Integrals	63
29.	Special Functions	39
30.	Partial Fraction Expansions	15
31.	Elementary and miscellaneous analysis	36
	16 Chapters of first Notebook	54

In the concluding part V, Berndt examines the unorganized pages in all the three notebooks of Ramanujan on continued fractions, alternative theories of elliptic functions, class invariants and singular moduli, explicit values of theta-functions, modular equations, infinite series, approximations and asymptotic expansions, etc. Berndt points out that *very few claims in this volume pertain to Ramanujan's published papers and problems.*

Ramanujan's Notebooks, Part V, Bruce C. Berndt ([VIII], 1997)[7]:

Ch.	Subject	# of Results
32.	Continued Fractions	73
33.	Ramanujan's Theories of Elliptic Functions to Alternative Bases	62
34.	Class Invariants and Singular Moduli	196
35.	Values of Theta-Functions	24
36.	Modular Equations and Theta-Function Identities in Notebook 1	87
37.	Infinite Series	53
38.	Approximations and Asymptotic Expansions	46
39.	Miscellaneous Results in the First Notebook	24

Many theorems communicated by Ramanujan to Hardy in his Jan.16, Feb. 27, 1913 letters are found in Chapters.22 – 31. In all, $759+605+834+491+565 = 3254$ theorems have been studied in the parts I–V by Prof. Berndt. Hardy had estimated that the notebooks of Ramanujan contained approximately 3000 – 4000 statements of theorems and besides editing one chapter of Ramanujan's second Notebook, in 1923, he tried to get them edited. His intentions have at long last been fulfilled by the tireless efforts of the indefatigable Bruce C. Berndt. For the motivated student of Mathematics, this

[7]The author thanks Prof. Berndt for readily communicating the Preface and the Contents pages of Part V, by e-mail, on request.

five part publication will provide a plethora of results for further studies. As stated elsewhere [XV], it is no exaggeration to say that as long as people do mathematics, the work of Ramanujan and the stupendous effort of Prof. Bruce C. Berndt in editing the Ramanujan notebooks will be appreciated.

R.P. Agarwal's work on the notebooks

Prof. Ratan P. Agarwal is the founder of a school of ordinary and basic hypergeometric series in India. In recent times, he has consolidated a part of his work in a publication entitled: *Resonance of Ramanujan's Mathematics* [82]. This two volume work concerns some of the results of Srinivasa Ramanujan (1887 - 1920), the renowned Indian Mathematician, recorded in his notebooks, which continue to be a perennial source of inspiration for generations of Mathematicians.

Volume I has five chapters. Chapter 1 concerns certain summation and asymptotic formulae for generalized (ordinary and basic) hypergeometric series. Chapter 2 is on Rogers-Ramanujan identities. Chapters 3 - 5 deal with certain definite Integrals of interest to Ramanujan.

Prof. Agarwal opines, like Hardy, that *a major part of Ramanujan's systematic work on ordinary hypergeometric series has been in the nature of a rediscovery of the work already done by previous authors* – the ${}_7F_6(1)$ summation formula of Dougall (1907) and particular cases of this sum, such as, Pfaff (1797) - Saalschütz (1890) theorem for the ${}_3F_2(1)$ and the Gauss and Kummer summation theorems for the ${}_2F_1(1)$ and the ${}_2F_1(-1)$, respectively, and Dixon's summation theorem for a well-poised ${}_3F_2(1)$. These and Ramanujan's entries in his notebooks concerning the deep results on the asymptotic relations for a finite number of terms of a variety of ordinary hypergeometric series, which center around the behaviour of the logarithmic solutions of the hypergeometric differential equation are presented in Chapter 1.

Chapter 2 deals with the derivations of the celebrated Rogers–Ramanujan identities[8], their combinatorial interpretations, their generalizations and their

[8]Refer chapter 2 of this work for a discussion about these famous identities.

applications in Statistical Mechanics for the solution of the two-dimensional hard-hexagonal model (by R.J. Baxter). Several proofs of these identities and the interesting mathematical developments have been described very well in this cogently written chapter.

The next three chapters of Volume I deal with some of the integrals studied by Ramanujan. Prof. Agarwal presents definite integrals associated with Fourier transforms in the second Notebook and self-reciprocal functions given in his 'lost' Notebook in Chapter 3. He has shown how some of Ramanujan's results on integrals in his Notebook can be established with the help of other entries of his

> *that every entry given in the notebooks by him has a purpose, although sometimes a particular entry may look out of context at the first instance. ... We have advocated elsewhere also that it may be interesting many a time to try to prove 'unproven' entry of Ramanujan with the help of some other entry given by him earlier or later on.*

Chapter 4 is focused on Ramanujan's Master Theorem for the evaluation of definite integrals, viz.:

$$\int_0^\infty x^{n-1}F(x)dx = \Gamma(n)\varphi(-n), \qquad \text{where } F(x) = \sum_{k=0}^{\infty} \frac{\varphi(k)(-x)^k}{k!},$$

in some neighbourhood of $x = 0$ and for n not necessarily a positive integer. The three Quarterly Reports submitted by Ramanujan to the Madras University, as a requirement stipulated by the University for awarding him the *First Research Scholarship in Mathematics*, contain various applications of this theorem to the *evaluation of a huge variety of integrals* and expansion formulae. The appropriate conditions of convergence for its validity have been provided by Hardy in his lectures contained in *Ramanujan* [IV]. The q-extension of the Master Theorem by Jackson (1951) and its extension to two variables by R.P. Agarwal (1974) are presented. The fundamental gamma and beta functions of mathematical analysis have been ingeniously extended by Ramanujan and these are related to the 'incomplete' gamma and beta functions (in which the upper limit of the integrals is finite and not ∞). The q-extensions of these works by several mathematicians including

Thomae (1879), F.H. Jackson (1904), R. Askey (1978), G.E. Andrews and R. Askey (1981), A. Verma and V.K. Jain (1992) are also included.

Chapter 5 is devoted to the study of integrals of the type:

$$\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\,dt$$

for particular values of a, b, c, d, evaluated by Ramanujan. Mordell (1933) classified these according to the values of the parameters and evaluated them in terms of Jacobi's theta and other related functions, using the method of complex contour integration. Professor Agarwal presents the general method of evaluation of different standard forms of this integral and relates them to those studied by Ramanujan, who used transform calculus to evaluate his integrals. Integrals associated with Riemann's zeta functions and elliptic modular relations are not discussed, as pointed out by the author himself. The chapter concludes with the mention of the example of a definite integral with fractional derivatives in the integrand stated in the third Quarterly Report. Ramanujan's formal deduction is based on his Master theorem and it agrees with the classical Liouville's definition of a fractional derivative of a function.

In Volume II, Prof. Agarwal presents a critical appraisal of Ramanujan's extensive and intriguing work on elliptic functions. Results pertaining to theta functions, partial theta functions, 'mock' theta functions (discovered by Ramanujan during the last year of his life on his return to India), as well as Lambert series and their relationship with elliptic functions, 'mock' theta functions and allied functions. While theta functions and partial theta functions are covered in chapter 1 of this volume and Lambert series and related functions are dealt with in chapter 5, the major part of this volume is concerned with the 'mock' theta function results of Ramanujan, presented in three extensive chapters (2, 3 and 4), contained in the 'Lost' Notebook, which was made available to the world in a facsimile edition, by Narosa, for the first time in the centenary year of Ramanujan's birth 1987 (though it was discovered in the estate of G.N. Watson by Prof. George E. Andrews of the Pennsylvania State University, in Spring 1976).

After giving the definitions of the four Jacobi theta functions and the 'false' theta functions (viz. theta functions with the 'wrong' signs for the terms) defined by L.J. Rogers, Prof. Agarwal introduces the Ramanujan notation for the theta function in terms of his symmetric function:

$$f(a, b) = \sum_{k=-\infty}^{\infty} (ab)^{k(k-1)/2}(a^k + b^k), \quad |ab| < 1.$$

In terms of this symmetric function, Ramanujan defines:

$$f(qe^{2iz}, qe^{-2iz}) = \theta_3(z, q), \text{ where } q = e^{i\pi\tau} \text{ and } |q| < 1.$$

Several general properties of the function $f(a, b)$ are then presented. Jacobi's triple product identity and other such identities due to D. Hickerson (1988) and G.E. Andrews and D. Hickerson (1991) are derived explicitly, due to their importance to simplify calculations in later chapters on 'mock' theta functions. A number of theta function expansions in the 'lost' notebook pertain to 'partial' theta functions. Some of these are stated and G.E. Andrews (1981) derived a general basic hypergeometric identity and showed the results of Ramanujan as special cases of it.

Ramanujan defined four third order mock theta functions and to this set three more were added by Watson. Watson used the transformations of basic hypergeometric series for obtaining new definitions for all the seven functions. Agarwal himself has contributed significantly to the theory of 'mock' theta functions by showing that most of the general identities of Andrews (referred above) belong to a very general class of basic hypergeometric transformations. His results are presented and those of Andrews deduced from them. Furthermore, Agarwal defines the third order mock theta functions in terms of a ${}_2\Phi_1$ basic hypergeometric series and deduces many of the mock theta function properties from the well-known properties of the ${}_2\Phi_1$. Ramanujan gave ten mock theta functions of order five, in two groups of five each, and three functions of order seven. He asserted that the members of each of the two groups of mock theta functions of order five are related amongst those belonging to the same group only, while the three mock theta functions of order seven are not related to each other. The works of Andrews, Agarwal and M. Gupta on the mock theta functions of order five are presented in detail. A. Gupta and Agarwal showed that mock theta functions of order five and

seven are defined through basic hypergeometric series of type ${}_3\Phi_2$ and ${}_4\Phi_3$. They have also made attempts to develop the theory of the above types of basic hypergeometric series to obtain new transformations and definitions for certain mock theta functions of orders five and seven. It is pointed out that the absence of a general transformation theory for the ${}_3\Phi_2(a, b, c; e, f; z)$ series [except when the argument z is q (the base parameter itself) or ef/abc], is the main reason why there exists no general transformation theory for mock theta functions of order five.

Observations of Agarwal on the relationship between the 'mock' theta functions and the basic hypergeometric series lead him to define:

> *the order of a mock theta function as* $(2r + 1)$ *if it is expressible in terms of a* ${}_{r+1}\Phi_r$ *series, on a single base* q^k, $k \leq r + 1$. *There may be in the definition of the mock theta function an additive term with* ${}_{r+1}\Phi_r$ *consisting of* θ*-products, which do not affect the order.*

While this is an intersting observation by itself, it is simplistic, to say the least. For, mystery still surrounds the incomplete work of Ramanujan in this area contributing to the belief that Ramanujan was working on a general theory of mock theta functions, whose preliminary results are the ones in the 'lost' notebook of his and that he was snatched away by fate before he could reveal his grand scheme!

This two volume work of Prof. Agarwal is a deep study of some of the work of Ramanujan on ordinary and basic hypergeometric series, on definite integrals and theta, partial theta and mock theta functions. It is very original in parts and has a minimum overlap with the comprehensive five part work on the notebooks of Ramanujan by Bruce C. Berndt[9]. This work presents, in as coherent a way as is possible, the work of Agarwal and his school of students and coworkers. It is a valuable supplement for the research scholar and for mathematicians and presents the author's understanding of the 'how' and 'why' of what Ramanujan did.

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[9]The author's reference to these Parts I to V as the Berndt Notebooks is unfortunate, especially in view of the significance of the Ramanujan notebooks and the 'Lost' notebook, which are the source materials for both Berndt and Agarwal.

Dyson on Ramanujan's notebooks

Freeman J. Dyson, the renowned Physicist ([XVI], p.25), concludes his article entitled: *A Walk through Ramanujan's Garden*, with the following advise:

> *In conclusion, I would like to urge all of you who are working in the many fields of mathematics which have been enriched by Ramanujan's ideas to go back to the source the collected papers and the notebooks. . . . The notebooks . . . are now appearing in a splendidly annotated version edited by Bruce Berndt [VIII]. The "lost" notebook is now accessible to us through the devoted labors of George Andrews ([53], [56]). When I started my walk through Ramanujan's garden 47 years ago, only the collected papers were available. A year after I chose Hardy and Wright's "Theory of Numbers" (Oxford, Clarendon Press, 1938) as a school prize, I won another prize. For the second prize I chose Ramanujan's collected papers. The collected papers have traveled with me from England to America and are still as fresh to-day as they were in 1940. Whenever I am angry or depressed, I pull down the collected papers from the shelf and take a quiet stroll in Ramanujan's garden. I recommend this therapy to all of you who suffer from headaches or jangled nerves. And Ramanujan's papers are not only a good therapy for headaches. They also are full of beautiful ideas which may help you to do more interesting mathematics.*

Certainly several of us will not find this therapy useful. Since Dyson, Erdös and Selberg were among the very best in their chosen fields, they were greatly benefited by an early introduction to the Collected Papers. The rest can derive the benefit of the extensive study of Ramanujan's work by Hardy [IV] and Berndt [VIII] to begin appreciating the prolific, original work of the mathematical genius in his published work and in his notebooks.

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A Note on "Mathematics Genius Srinivasa Ramanujan, FRS "

By Dr.S.Sridhar, Ph.D.
Director, R&D
SKR EnggCollege Chennai
drssridhar@yahoo.com

A Brief

Born 22 December 1887 Erode, T.Nadu

Died 26 April 1920 (aged 32) Chetput, Madras,

Residence Kumbakonam Nationality Indian

Fields Mathematics

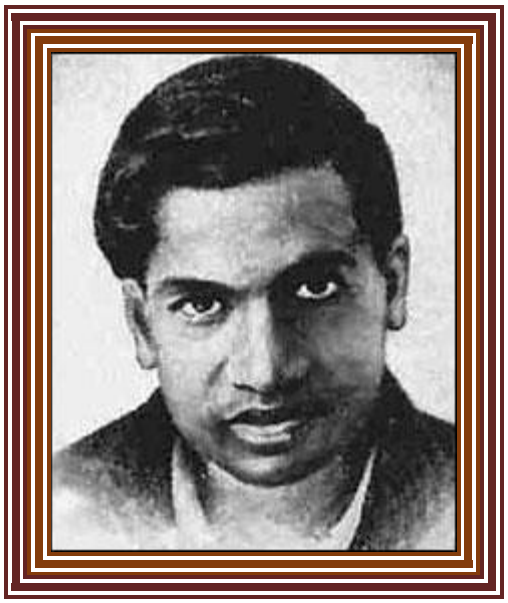
Alma mater Government Arts College
Pachaiyappa's College

Academic advisors G. H. Hardy,
J. E. Littlewood

Known for

Landau–Ramanujan constant
Mock theta functions
Ramanujan conjecture
Ramanujan prime
Ramanujan–Soldner constant
Ramanujan theta function
Ramanujan's sum
Rogers–Ramanujan identities
Ramanujan's master theorem

Influences G. H. Hardy



Abstract

Srinivasa Ramanujan *FRS (Fellow of Royal Society)(22 December 1887 – 26 April 1920) was an Indian mathematician , with almost no formal training in pure mathematics, made extraordinary contributions to mathematical analysis, number theory, infinite series, and continued fractions. Living in India with no access to the larger mathematical community, which was centered in Europe at the time, Ramanujan developed his own mathematical research in isolation. As a result, he sometimes rediscovered known theorems in addition to producing new work. Ramanujan was said to be a natural genius by the English mathematician G.H. Hardy, in the same league as mathematicians like Euler and Gauss.*

Born in a poor Brahmin family, Ramanujan's introduction to formal mathematics began at age 10. He demonstrated a natural ability, and was given books on advanced trigonometry written by S. L. Loney that he mastered by the age of 12; he even discovered theorems of his own, and re-discovered Euler's identity independently. He demonstrated unusual mathematical skills at school, winning accolades and awards. By 17, Ramanujan had conducted his own mathematical research on Bernoulli numbers .

Ramanujan received a scholarship to study at Government College in Kumbakonam, but lost it when he failed his non-mathematical coursework. He joined another college to pursue independent mathematical research, working as a clerk in the Accountant-General's office at the Madras Port Trust Office to support himself. In 1912–1913, he sent samples of his theorems to three academics at the University of Cambridge. G. H. Hardy, recognizing the brilliance of his work, invited Ramanujan to visit and work with him at Cambridge. He became a Fellow of the Royal Society and a Fellow of Trinity College, Cambridge. Ramanujan died of illness, malnutrition, and possibly liver infection in 1920 at the age of 32.

During his short lifetime, Ramanujan independently compiled nearly 3900 results (mostly identities and equations). Most of his claims have now been proven correct. He stated results that were both original and highly unconventional, such as the Ramanujan prime and the Ramanujan theta function, and these have inspired a vast amount of further research. However, the mathematical mainstream has been rather slow in absorbing some of his major discoveries. The Ramanujan Journal, an international publication, was launched to publish work in all areas of mathematics influenced by his work.

In December 2011, in recognition of his contribution to mathematics, the Government of India declared that Ramanujan's birthday (22 December) should be celebrated every year as National Mathematics Day, and also declared 2012 the National Mathematical Year.

Early Education & Personal life

Ramanujan was born on 22 December 1887 in Erode, Tamil nadu at the residence of his maternal grandparents. His father, K. Srinivasa Iyengar, worked as a clerk in a sari shop and hailed from the district of Thanjavur. His mother, Komalatammal, was a housewife and also sang at a local temple. They lived in Sarangapani Street in a traditional home in the town of Kumbakonam. In December 1889, Ramanujan had smallpox and recovered. He moved with his mother to her parents' house in Kanchipuram, near Chennai.

Since Ramanujan's father was at work most of the day, his mother took care of him as a child. From her, he learned about tradition and puranas. He learned to sing religious songs, to attend pujas at the temple and particular eating habits – all of which are part of Brahmin culture. Just before the age of 10, in November 1897, he passed his primary examinations in English, Tamil, geography and arithmetic. With his scores, he stood first in the district. That year, Ramanujan entered Town Higher Secondary School where he encountered formal mathematics for the first time.

By age 11, he had exhausted the mathematical knowledge of two college students who were lodgers at his home. He was later lent a book on advanced trigonometry written by S. L. Loney. He completely mastered this book by the age of 13 and discovered sophisticated theorems on his own. By 14, he was receiving merit certificates and academic awards which continued throughout his school career and also assisted the school in the logistics of assigning its 1200 students (each with their own needs) to its 35-odd teachers. He completed mathematical exams in half the allotted time, and showed a familiarity with geometry and infinite series. Ramanujan was shown how to solve cubic equations in 1902 and he went on to find his own method .

In 1903 when he was 16, Ramanujan obtained from a friend a library-loaned copy of a book by G. S. Carr. The book was titled *A Synopsis of Elementary Results in Pure and Applied Mathematics* and was a collection of 5000 theorems. Ramanujan reportedly studied the contents of the book in detail. The book is generally acknowledged as a key element in awakening the genius of Ramanujan. The next year, he had independently developed and investigated the Bernoulli numbers and had calculated Euler's constant up to 15 decimal places.

When he graduated from Town Higher Secondary School in 1904, Ramanujan was awarded the K. Ranganatha Rao prize for mathematics by the school's headmaster, Krishnaswami Iyer who introduced Ramanujan as an outstanding student who deserved scores higher than the maximum possible marks. He received a scholarship to study at Government Arts College, Kumbakonam, However, Ramanujan was so intent on studying mathematics that he could not focus on any other subjects and failed most of them, losing his scholarship in the process. He later enrolled at Pachaiyappa's College in Madras. He again excelled in mathematics but performed poorly in other subjects such as physiology. Ramanujan failed his Fine Arts degree exam in December 1906 and again a year later. Without a degree, he left college and continued to pursue independent research in mathematics. At this point in his life, he lived in extreme poverty and was suffering from starvation.

Adulthood in India

On 14 July 1909, Ramanujan was married to a nine-year old bride, Janaki Ammal (21 March 1899 - 13 April 1994. After the marriage, Ramanujan developed a hydrocele testis. His family did not have the money for the operation, but in January 1910, a doctor volunteered to do the surgery for free.

After his successful surgery, Ramanujan searched for a job. He stayed at friends' houses while he went door to door around the city of Chennai looking for a clerical position. To make some money, he tutored some students at Presidency College who were preparing for their exam.

Attention from mathematicians

Ramanujan met deputy collector V. Ramaswamy Aiyer, who had recently founded the Indian Mathematical Society . Ramanujan, wishing for a job at the revenue department where Ramaswamy Aiyer worked, showed him his mathematics notebooks. As Ramaswamy Aiyer later recalled:

I was struck by the extraordinary mathematical results contained in it [the notebooks]. I had no mind to smother his genius by an appointment in the lowest level as clerk in the revenue department.

Ramaswamy Aiyer sent Ramanujan, with letters of introduction, to his mathematician friends in Madras.– Some of these friends looked at his work and gave him letters of introduction to R. Ramachandra Rao, the district collector for Nellore and the secretary of the Indian Mathematical Society. Ramachandra Rao was impressed by Ramanujan's research but doubted that it was actually his own work. Ramanujan mentioned a correspondence he had with Professor Saldhana, a notable Bombay mathematician, in which Saldhana expressed a lack of understanding for his work but concluded that he was not a phony. Ramanujan's friend, C. V. Rajagopalachari, persisted with Ramachandra Rao and tried to clear any doubts over Ramanujan's academic integrity. Rao agreed to give him another chance, and he listened as Ramanujan discussed elliptic integrals, hypergeometric series, and his theory of divergent series, which Rao said ultimately "converted" him to a belief in Ramanujan's mathematical brilliance. When Rao asked him what he wanted, Ramanujan replied that he needed some work and financial support. Rao consented and sent him to Madras. He continued his mathematical research with Rao's financial aid taking care of his daily needs. Ramanujan, with the help of

Ramaswamy Aiyer, had his work published in the *Journal of Indian Mathematical Society*.

One of the first problems he posed in the journal was:

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}.$$

He waited for a solution to be offered in three issues, over six months, but failed to receive any. At the end, Ramanujan supplied the solution to the problem himself. On page 105 of his first notebook, he formulated an equation that could be used to solve the infinitely nested radicals problem.

$$x+n+a=\sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{\cdots}}}$$

Using this equation, the answer to the question posed in the *Journal* was simply 3. Ramanujan wrote his first formal paper for the *Journal* on the properties of Bernoulli numbers. One property he discovered was that the denominators of the fractions of Bernoulli numbers were always divisible by six. He also devised a method of calculating B_n based on previous Bernoulli numbers. One of these methods went as follows:

Mr. Ramanujan's methods were so novel and his presentation so lacking in clearness and precision, that the ordinary mathematical reader, unaccustomed to such intellectual gymnastics, could hardly follow him.

Ramanujan later wrote another paper and also continued to provide problems in the *Journal*. In early 1912, he got a temporary job in the Madras Accountant General's office, with a salary of 20 rupees per month. He lasted for only a few weeks. Toward the end of that assignment he applied for a position under the Chief Accountant of the Madras Port Trust.

Attached to his application was a recommendation from E. W. Middlemast, a mathematics professor at the Presidency College, who wrote that Ramanujan was "a young man of quite exceptional capacity in Mathematics". Three weeks after he had applied, on 1 March, Ramanujan learned that he had been accepted as a Class III, Grade IV accounting clerk, making 30 rupees per month. At his office, Ramanujan easily and quickly completed the work he was given, so he spent his spare time doing mathematical research. Ramanujan's boss, Sir Francis Spring, and S. Narayana Iyer, a colleague who was also treasurer of the Indian Mathematical Society, encouraged Ramanujan in his mathematical pursuits.

Contacting English mathematicians

On the spring of 1913, Narayana Iyer, Ramachandra Rao and E. W. Middlemast tried to present Ramanujan's work to British mathematicians. One mathematician, M. J. M. Hill of University College London, commented that Ramanujan's papers were riddled with holes. He said that although Ramanujan had "a taste for mathematics, and some ability", he lacked the educational background and foundation needed to be accepted by mathematicians. Although Hill did not offer to take Ramanujan on as a student, he did give thorough and serious professional advice on his work. With the help of friends, Ramanujan drafted letters to leading mathematicians at Cambridge University.

On 16 January 1913, Ramanujan wrote to G. H. Hardy. Coming from an unknown

mathematician, the nine pages of mathematics made Hardy initially view Ramanujan's manuscripts as a possible "fraud". Hardy recognised some of Ramanujan's formulae but others "seemed scarcely possible to believe".

Hardy was also impressed by some of Ramanujan's other work relating to infinite series:

$$1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1\times 3}{2\times 4}\right)^3-13\left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^3+\cdots=\frac{2}{\pi}$$

$$1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1\times 5}{4\times 8}\right)^4+25\left(\frac{1\times 5\times 9}{4\times 8\times 12}\right)^4+\cdots=\frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}\Gamma^2\left(\frac{3}{4}\right)}.$$

The second one was new to Hardy, and was derived from a class of functions called a hypergeometric series which had first been researched by Leonhard Euler and Carl Friedrich Gauss. After he saw Ramanujan's theorems on continued fractions on the last page of the manuscripts, Hardy commented that the "[theorems] defeated me completely; I had never seen anything in the least like them before". He figured that Ramanujan's theorems "must be true, because, if they were not true, no one would have the imagination to invent them". Hardy asked a colleague, J. E. Littlewood, to take a look at the papers. Littlewood was amazed by the mathematical genius of Ramanujan. After discussing the papers with Littlewood, Hardy concluded that the letters were "certainly the most remarkable I have received" and commented that Ramanujan was "a mathematician of the highest quality, a man of altogether exceptional originality and power".

On 8 February 1913, Hardy wrote a letter to Ramanujan, expressing his interest for his work. Hardy also added that it was "essential that I should see proofs of some of your assertions". Before his letter arrived in Madras during the third week of February, Hardy contacted the Indian Office to plan for Ramanujan's trip to Cambridge. Ramanujan refused to leave his country to "go to a foreign land". Meanwhile, Ramanujan sent a letter packed with theorems to Hardy, writing, "I have found a friend in you who views my labour sympathetically."

Ramanujan continued to submit papers to the *Journal of the Indian Mathematical Society*. In one instance, Narayana Iyer submitted some theorems of Ramanujan on summation of series to the above mathematical journal adding "The following theorem is due to S. Ramanujan, the mathematics student of Madras University".

Hardy's correspondence with Ramanujan soured after Ramanujan refused to come to England. Apparently, Ramanujan's mother had a vivid dream in which the family Goddess Namagiri commanded her "to stand no longer between her son and the fulfilment of his life's purpose".

Life in England

Ramanujan (centre) with other scientists at Trinity College

Whewell's Court, Trinity College, Cambridge

Ramanujan arrived in London on 14 April, with E. H. Neville waiting for him with a car. Four days later, Neville took him to his house on Chesterton Road in Cambridge. Ramanujan immediately began his work with Littlewood and Hardy. After six weeks, Ramanujan moved out of Neville's house and took up residence on Whewell's Court, just a five-minute walk from Hardy's room.[74] Hardy and Ramanujan began to take a look at Ramanujan's notebooks. Hardy had already received 120 theorems from Ramanujan in the first two letters, but there were many more results and theorems to be found in the notebooks. Ramanujan left a deep impression on Hardy and Littlewood. Littlewood commented, "I can believe that he's at least a Jacobi", while Hardy said he "can compare him only with Euler or Jacobi."

Ramanujan spent nearly five years in Cambridge collaborating with Hardy and Littlewood and published a part of his findings there. Hardy and Ramanujan had highly contrasting personalities. Their collaboration was a clash of different cultures, beliefs and working styles. While in England, Hardy tried his best to fill the gaps in Ramanujan's education without interrupting his spell of inspiration.

Ramanujan was awarded a B.A. degree by research (this degree was later renamed PhD) in March 1916 for his work on highly composite numbers, the first part of which was published as a paper in the *Proceedings of the London Mathematical Society*. The paper was over 50 pages with different properties of such numbers proven. Hardy remarked that this was one of the most unusual papers seen in mathematical research at that time and that Ramanujan showed extraordinary power in handling it. On 6 December 1917, he was elected to the

London Mathematical Society. He became a Fellow of the Royal Society in 1918, becoming the second Indian to do so and he was one of the youngest Fellows in the history of the Royal Society. He was elected "for his investigation in Elliptic functions and the Theory of Numbers." On 13 October 1918, he became the first Indian to be elected a Fellow of Trinity College, Cambridge.

Illness and return to India

Plagued by health problems throughout his life, living in a country far away from home, and deeply involved with his mathematics, Ramanujan's health worsened in England, by stress and by the scarcity of vegetarian food during the First World War. He was suffering from tuberculosis and a severe vitamin deficiency .

Ramanujan returned to Kumbakonam, Madras Presidency in 1919 and died soon thereafter at the age of 32. His widow, S. Janaki Ammal, lived in Chennai (formerly Madras) until her death in 1994.

Mathematical achievements

In mathematics, there is a distinction between having an insight and having a proof. Ramanujan's talent suggested a group of formulae that could then be investigated in depth later. It is said that Ramanujan's discoveries are unusually rich . As a by-product, new directions of research were opened up. Examples of the most interesting of these formulae include the infinite series for π, one of which is given below

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

One of his remarkable capabilities was the rapid solution for problems. He was sharing a room with P. C. Mahalanobis who had a problem, "Imagine that you are on a street with houses marked 1 through n. There is a house in between (x) such that the sum of the house numbers to left of it equals the sum of the house numbers to its right. If n is between 50 and 500, what are n and x?" This is a bivariate problem with multiple solutions. Ramanujan thought about it and gave the answer with a twist: He gave a continued fraction. The unusual part was that it was the solution to the whole class of problems. Mahalanobis was astounded and asked how he did it. "It is simple. The minute I heard the problem, I knew that the answer was a continued fraction. Which continued fraction, I asked myself. Then the answer came to my mind," Ramanujan replied.

In 1918, Hardy and Ramanujan studied the partition function $P(n)$ extensively and gave a non-convergent asymptotic series that permits exact computation of the number of partitions of an integer. He discovered mock theta functions in the last year of his life. For many years these functions were a mystery, but they are now known to be the holomorphic parts of harmonic weak Maass forms.

The Ramanujan conjecture

Main article: Ramanujan–Petersson conjecture

Although there are numerous statements that could bear the name *Ramanujan conjecture*, there is one statement that was very influential on later work. In particular, the connection of this conjecture with conjectures of André Weil in algebraic geometry opened up new areas of research.

That Ramanujan conjecture is an assertion on the size of the tau function,

Ramanujan's notebooks

Ramanujan recorded the bulk of his results in four notebooks of loose leaf paper. These results were mostly written up without any derivations. This is probably the origin of the misperception that Ramanujan was unable to prove his results and simply thought up the final result directly.

This style of working may have been for several reasons. Since paper was very expensive, Ramanujan would do most of his work and perhaps his proofs on slate, and then transfer just the results to paper. Using a slate was common for mathematics students in the Madras Presidency at the time.

The first notebook has 351 pages with 16 somewhat organized chapters and some unorganized material. The second notebook has 256 pages in 21 chapters and 100 unorganised pages, with the third notebook containing 33 unorganised pages. The results in his notebooks inspired numerous papers by later mathematicians trying to prove what he had found. Hardy himself created papers exploring material from Ramanujan's work as did G. N. Watson, A fourth notebook with 87 unorganised pages, the so-called "lost notebook", was rediscovered in 1976 by George Andrews.

Notebooks 1, 2 and 3 were published as a two volume set in 1957 by the Tata Institute of Fundamental Research (TIFR), Mumbai, India. This was a photocopy edition of the original manuscripts, in his own handwriting.

In December 2011, as part of Ramanujan's 125th birth centenary celebrations, TIFR republished the notebooks in a colored two volume collector's edition. These were produced from scanned and microfilmed images of the original manuscripts by expert archivists of Roja Muthiah Research Library, Chennai.

Ramanujan–Hardy number 1729

Main article: 1729 (number)

The number 1729 is known as the Hardy–Ramanujan number after a famous British mathematician G. H. Hardy regarding a visit to the hospital to see Ramanujan. In Hardy's words:

> " I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable . "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." "

The two different ways are

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Generalizations of this idea have created the notion of "taxicab numbers".

Other mathematicians' views of Ramanujan

Hardy said : " Here was a man who could work out modular equations and theorems... to orders unheard of, whose mastery of continued fractions was... beyond that of any mathematician in the world, who had found for himself the functional equation of the

zeta function and the dominant terms of many of the most famous problems in the analytic theory of numbers; and yet he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was.". He also stated that he had "never met his equal, and can compare him only with Euler or Jacobi."

Quoting K. Srinivasa Rao , "As for his place in the world of Mathematics, we rate mathematicians on the basis of pure talent on a scale from 0 to 100, Hardy gave himself a score of 25, J.E. Littlewood 30, David Hilbert 80 and Ramanujan 100.'"

Professor Bruce C. Berndt of the University of Illinois, during a lecture at IIT Madras in May 2011, stated that over the last 40 years, as nearly all of Ramanujan's theorems have been proven right, there had been a greater appreciation of Ramanujan's work and brilliance. Further, he stated Ramanujan's work was now pervading many areas of modern mathematics and physics.

In his book *Scientific Edge*, noted physicist Jayant Narlikar spoke of "Srinivasa Ramanujan, discovered by the Cambridge mathematician Hardy, whose great mathematical findings were beginning to be appreciated from 1915 to 1919. His achievements were to be fully understood much later, well after his untimely death in 1920. "

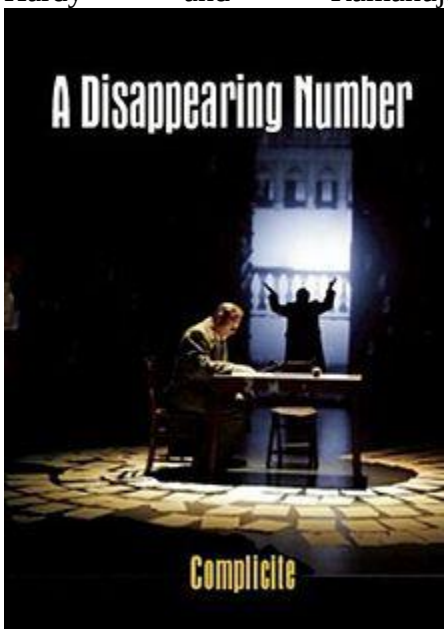
During his lifelong mission in educating and propagating mathematics among the school children in India, Nigeria and elsewhere, P.K. Srinivasan has continually introduced Ramanujan's mathematical works.

Recognition

- Ramanujan's home state of Tamil Nadu celebrates 22 December (Ramanujan's birthday) as 'State IT Day', memorializing both the man and his achievements, as a native of Tamil Nadu. A stamp picturing Ramanujan was released by the Government of India in 1962 – the 75th anniversary of Ramanujan's birth – commemorating his achievements in the field of number theory and a new design was issued on December 26, 2011, by the India Post.
- Since the Centennial year of Ramanujan, every year 22 Dec, is celebrated as Ramanujan Day by the Government Arts College, Kumbakonam where he had studied and later dropped out.
- Ramanujan's work and life are celebrated on 22 December at The Indian Institute of Technology (IIT), Madras in Chennai.
- A prize for young mathematicians from developing countries has been created in the name of Ramanujan by the International Centre for Theoretical Physics (ICTP), in cooperation with the International Mathematical Union, who nominate members of the prize committee. The Shanmugha Arts, Science, Technology & Research Academy (SASTRA), based in the state of Tamil Nadu in South India, has instituted the SASTRA Ramanujan Prize of $10,000 to be given annually to a mathematician not exceeding the age of 32 for outstanding contributions in an area of mathematics influenced by Ramanujan. The age limit refers to the years Ramanujan lived, having nevertheless still achieved many accomplishments. This prize has

been awarded annually since 2005, at an international conference conducted by SASTRA in Kumbakonam, Ramanujan's hometown, around Ramanujan's birthday, 22 December.

- On the 125th anniversary of his birth, India declared the birthday of Ramanujan, December 22, as 'National Mathematics Day.' The declaration was made by Dr. Manmohan Singh in Chennai on December 26, 2011. Dr Manmohan Singh also declared that the year 2012 would be celebrated as the National Mathematics YearA Disappearing Number
- A film, based on the book *The Man Who Knew Infinity: A Life of the Genius Ramanujan* by Robert Kanigel, is being made by Edward Pressman and Matthew Brown with R. Madhavan playing Ramanujan.
- Another international feature film on Ramanujan's life was announced in 2006 as due to begin shooting in 2007. It was to be shot in Tamil Nadu state and Cambridge and be produced by an Indo-British collaboration and co-directed by Stephen Fry and Dev Benegal.
- A play, *First Class Man* by Alter Ego Productions, was based on David Freeman's *First Class Man*. The play is centred around Ramanujan and his complex and dysfunctional relationship with Hardy. Like the book and play it is also titled *The First Class Man*; the film's scripting has been completed and shooting is being planned from 2012.
- *A Disappearing Number* is a recent British stage production that explores the relationship between Hardy and Ramanujan.



- The novel *The Indian Clerk* by David Leavitt explores in fiction the events following Ramanujan's letter to Hardy.
- On 22 March 1988, the PBS Series *Nova* aired a documentary about Ramanujan, "The Man Who Loved Numbers" (Season 15, Episode 9).

Selected publications by Ramanujan

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- S. Ramanujan (1957). *Notebooks (2 Volumes)*. Bombay: Tata Institute of

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Srinivasa Ramanujan (1887–1920)
The Centenary of a Remarkable Mathematician

Adrian Rice, Randolph-Macon College, Virginia, USA

This month marks the centenary of the death of one of the most remarkable mathematicians of the 20th century. The enigmatic Indian mathematician Srinivasa Ramanujan was perhaps one of the most original mathematicians of all time. In a career that only lasted around ten years, he produced hundreds of highly innovative results in several areas of pure mathematics, particularly number theory and analysis. After his death, his notebooks and unpublished results inspired decades of research by succeeding mathematicians, the impact of which is still being felt in mathematics today. What follows is based in large part on the work of these scholars, particularly G.H. Hardy [1,2], George Andrews [3], Bruce Berndt [3–5] and Ramanujan's biographer, Robert Kanigel [6].

Srinivasa Ramanujan was born to a modest Brahmin family on 22 December 1887 in the town of Erode in Tamil Nadu, southern India. Due to his father's heavy work schedule, the boy formed a close relationship with his mother, and it was from her that he acquired his religious beliefs and adherence to specific customs, particularly his strict vegetarianism. Performing well at primary school, he passed exams in English, Tamil, geography and arithmetic at the age of 10 with the best scores in his school district. After beginning secondary level mathematics, by his early teens he was investigating and discovering his own independent results. For example, by the age of 15, having been taught the method for solving cubic equations, he had created his own algorithm to solve the quartic.

The biggest influence on his mathematical development appears to have been his acquisition, at the age of 16, of a copy of *A Synopsis of Elementary Results in Pure and Applied Mathematics* by a certain G.S. Carr [7]. This was a large compendium of hundreds of mathematical formulae and theorems, listed thematically with little to no commentary or motivation. Although the book did contain brief derivations of some of the results, such demonstrations were minimal and the overall content was terse and dry. Nevertheless, its content appears to have had a profound effect on Ramanujan, stimulating a fascination with formulae and symbolic manipulation that never left him, and influencing his preferred style of mathematical presentation.

Upon graduation from secondary school in 1904, he was awarded a scholarship to study at the Government College in Kumbakonam, also in Tamil Nadu. However, his obsession with his own mathematical research resulted in his failing important exams in most of his other subjects of study and consequently losing his scholarship. His health was also proving problematic, with absences from classes caused by illness occurring with some frequency. Later enrolling at Pachaiyappa's College, Madras (now called Chennai), he failed to obtain a degree in 1907 for essentially the same reasons as before. Forced to leave college at the age of 20, with no qualifications, poor health and poorer career prospects, the next few years were of considerable hardship for Ramanujan as he lived in poverty while trying to forge ahead with his research in mathematics.

As early as 1904, Ramanujan had begun to produce mathematical research of substantial sophistication. Of course, having

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Figure 1: Srinivasa Ramanujan (1887–1920)

no contact with any active mathematical researchers and being essentially self-taught, he had no knowledge of contemporary research topics, so would largely pursue his own ideas, often inspired by formulae and techniques from Carr's *Synopsis*. He was particularly fascinated by infinite series and processes. An early example of a problem he solved was to find the value of

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}.$$

Ramanujan's solution noted that, since $n\sqrt{1+(n+1)(n+3)} = n(n+2)$, if we let $f(n) = n(n+2)$, then

$$\begin{aligned} f(n) &= n\sqrt{1+f(n+1)} \\ &= n\sqrt{1+(n+1)\sqrt{1+f(n+2)}} \\ &= n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+f(n+3)}}}, \end{aligned}$$

and so on, so that, ultimately,

$$n(n+2) = n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\cdots}}}}.$$

Letting $n = 1$ then revealed that

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}} = 3.$$

In July 1909, Ramanujan was married to Janaki Ammal in an arranged marriage that took place when Janaki was just 10 years old. Although such a practice was not unusual at the time, Ramanujan's wife did not actually live with him for the first three years of their marriage, moving in with him and his mother in 1912. Meanwhile his reputation as a mathematician began, very slowly, to grow. Following his introduction in 1910 to V. Ramaswamy Aiyer, the founder of the Indian Mathematical Society, Ramanujan began to publish his results in the Society's *Journal*. His first paper, 'Some properties of Bernoulli's numbers' [8], stemmed from research undertaken in his teens, when he had discovered and developed the Bernoulli numbers in complete ignorance of any prior research on the subject.

But he continued to struggle to find gainful employment to support his family. After combinations of temporary clerical work and private tutoring, he managed to secure a position as a clerk in the Madras Port Trust. His growing circle of mathematical friends in the Madras area became convinced that his work should be brought to the attention of mathematicians in Britain. Early attempts were disappointing, however. A letter to M.J.M. Hill, professor of mathematics at University College London, received a faintly encouraging, but largely patronising, response, while letters to Cambridge mathematicians E.W. Hobson and H.F. Baker received no reply. It was then, in January 1913, that Ramanujan wrote one of the most famous letters in the history of mathematics.

Its recipient was G.H. Hardy, a lecturer at Trinity College, Cambridge, who was one of Britain's foremost pure mathematicians. On reading the multi-page letter crammed with dozens of intricate formulae and theorems from Ramanujan's notebooks, Hardy's first reaction could have been to dismiss everything as the work of a fraud or a crank. But a close examination of its content by Hardy and his Cambridge colleague J.E. Littlewood revealed a host of amazing results. These Hardy divided into three categories. Firstly, there were theorems that, unbeknown to Ramanujan, were already known, such as the integral formula:

$$\int_0^a \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{2}\pi^{1/2} - \cfrac{\mathrm{e}^{-a^2}}{2a + \cfrac{1}{a + \cfrac{2}{2a + \cfrac{3}{a + \cfrac{4}{2a + \cdots}}}}}.$$

Secondly, there were results that, while new, were interesting rather than important, for example:

$$\int_0^\infty \frac{\cos \pi x}{\Gamma^2(\alpha + x)\Gamma^2(\alpha - x)}\,\mathrm{d}x = \frac{1}{4\Gamma(2\alpha - 1)\Gamma^2(\alpha)},$$

where $\alpha > 1/2$ and $\Gamma(z)$ is the gamma function. And finally, there were entirely original results that were simply astonishing, such as:

$$\cfrac{1}{1 + \cfrac{\mathrm{e}^{-2\pi}}{1 + \cfrac{\mathrm{e}^{-4\pi}}{1 + \cdots}}} = \left[\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right] \mathrm{e}^{\frac{2}{5}\pi}.$$

These formulae, Hardy later wrote [4, part II, p. 103],

> … defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.

It quickly became apparent that Ramanujan was a mathematician of exceptional ability, and Hardy soon began to consider ways in which he could be brought to Britain to receive formal training and an advanced degree from Cambridge. Ramanujan, however, was initially unwilling to consider leaving his wife and family; so, as a short-term measure, his mathematical friends in India applied, successfully, for a temporary research scholarship on his behalf at the University of Madras, which allowed him to concentrate on his mathematical research full time while supporting his family financially. But there was still the matter of persuading him to make the long sea journey to England. This he refused to do for a number of reasons, some of them religious and some related to family obligations. But after a year of wrangling, these obstacles were gradually overcome and, with his parents' eventual consent, he finally agreed to go, setting sail for England on 17 March 1914.

On his arrival in Cambridge the following month, he immediately set to work with Hardy and Littlewood. Not surprisingly, however, they soon found that their mathematical methodologies were profoundly different. Hardy and Littlewood, being strict analysts, were insistent on absolute rigour and formal proofs, while Ramanujan was content to rely on intuition and inductive experimentation. Indeed, as Littlewood later wrote [9, p.88],

> He was not interested in rigour … [and] the clear-cut idea of what is *meant* by a proof … he perhaps did not possess at all.

Equally inevitably, given Ramanujan's erratic higher education, there were also substantial gaps in his mathematical knowledge. It was a source of constant surprise to Hardy that, despite Ramanujan's expertise in elliptic functions (almost certainly self-taught), he had no experience at all in the theory of complex functions. Hardy thus sought to remedy the defects in Ramanujan's formal training without dampening his enthusiasm and wild imagination. It proved to be a challenge.

Adapting to life in Britain was equally challenging for Ramanujan. In addition to the colder and damper climate, the practice of vegetarianism was not nearly as easy in early 20th century Cambridge as it is today. He cooked his own simple meals in his room in Trinity College, and thus did not eat with the rest of the college community as would usually have been expected. The outbreak of the First World War in August 1914, not long after his arrival in Britain, made things harder still, as rationing made it increasingly difficult to maintain a healthy diet with the limited variety of produce on offer. The war also resulted in many of Cambridge's best mathematicians, including Littlewood, being called away for war service, further adding to Ramanujan's sense of isolation.

Nevertheless, Ramanujan's mathematical output was rapid. Within months he was publishing papers in British journals, including a 60-page paper [10] in the *Proceedings of the London Mathematical Society* in 1915. This was on a subject of his own creation called 'highly composite numbers'. Whereas number theorists are often largely concerned with properties and attributes of prime numbers, Ramanujan considered integers 'whose number of divisors exceeds that of all its predecessors'

and deduced a startling array of results about them. It was largely on the basis of this work that, in March 1916, he was awarded the degree of Bachelor of Science by Research (renamed PhD in 1920) by the University of Cambridge.

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Figure 2: G.H. Hardy (1877–1947)

Before long, Ramanujan and Hardy were collaborating on joint research and much of 1916 was spent working on their most famous collaboration: their joint paper on partitions [11]. The partition number $p(n)$ represents the number of ways that a positive integer n can be written as a sum of positive integers where the order of addition does not matter. So for instance, the integer $n = 4$ can be written in five different ways, namely,

$$\begin{aligned} 4 &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1, \end{aligned}$$

meaning that $p(4) = 5$. But the question quickly becomes more complicated. For example, $p(15) = 176$, and $p(34) = 12\,310$. What then is $p(100)$ or $p(200)$?

After months of effort, Ramanujan and Hardy produced an answer to this question in the form of one of the most staggering formulas in the whole of mathematics. If

$$A_q(n) = \sum_{p=1}^{q} \omega_{p,q}\, \mathrm{e}^{-2np\pi \mathrm{i}/q}$$

and

$$\phi(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \left[\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{\exp(a\lambda_z/q)}{\lambda_z} \right) \right]_{z=n}$$

where $p, q \in \mathbb{Z}^+$, $(p,q) = 1$, $a = \pi\sqrt{2/3}$, $\lambda_z = \sqrt{z - 1/24}$ and $\omega_{p,q}$ is a $24q$-th root of unity, then, for some $\alpha \in \mathbb{Z}^+$,

$$p(n) = \sum_{q=1}^{[\alpha\sqrt{n}]} A_q(n)\phi(n) + O(n^{-1/4}).$$

It is really remarkable to get a result with an error of $O(n^{-\frac{1}{4}})$ which is as good as that with any higher powers of n in this problem. Let c_s be the s-series which is $O(s)$ and $\Omega(1)$. Suppose now that s is about $\beta\sqrt{n}/\log n$ where $\beta = 4\pi\sqrt{2/3}$. Then

$$O\sum_{O(s)}^{O(s)} \frac{|c_s|\sqrt{s}}{sn} \mathrm{e}^{(\pi/s)\sqrt{(2n/3)}} = O(s\sqrt{s}n^{-\frac{3}{4}}) = O(\log n)^{-\frac{3}{2}}.$$

It would therefore follow from your arguments that the error by taking about $\beta\sqrt{n}/\log n$ terms is $O(\log n)^{-\frac{3}{2}}$. Again suppose that s is about $\alpha\sqrt{n}/\log n$ where $\alpha = (4\pi/5)\sqrt{2/3}$, then

$$\frac{|c_s|\sqrt{s}}{sn} \mathrm{e}^{(\pi/s)\sqrt{(2n/3)}} = \Omega\left(\frac{n^{\frac{1}{4}}}{\sqrt{s}}\right) = \Omega\left(\sqrt{\log n}\right).$$

It therefore appears that, in order that $p(n)$ may be the nearest integer to the approximate sum, s need not be taken beyond $\beta\sqrt{n}/\log n$ and cannot be taken below $\alpha\sqrt{n}/\log n$ …

Figure 3: Extract from a postcard sent by Ramanujan to Hardy as their work on partitions neared completion [5, p. 141].

They then proceeded to show that their formula was capable of producing results of unprecedented accuracy. For $n = 100$, it gave an output of 190 569 291.996, with the actual partition number being 190 569 292. For $n = 200$, the result was just as dramatic: their formula gave 3 972 999 029 388.004, with an error of just 0.004. Hardy and Ramanujan's formula displayed a breathtaking blend of mathematical ideas and influences. Fundamentally, it was a fusion of Ramanujan's dazzling powers of formulaic intuition with Hardy's mastery of the tools of analytic function theory. But without the intuitive genius of Ramanujan, Hardy would never have formulated such an astonishing result; and without Hardy, Ramanujan would never have been able to prove it. Littlewood said [9, p. 90],

> We owe the theorem, to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him.

Ramanujan's health had never been strong, but by May 1917 he was seriously ill, with his condition no doubt exacerbated by the British weather and the difficulty of maintaining adequate nutrition in wartime. He was diagnosed with tuberculosis and severe vitamin deficiency, although recent analysis has concluded that he may have been suffering from hepatic amoebiasis, a complication arising from previous attacks of dysentery. He spent much of the year in various nursing homes, but his mathematical output, although reduced, remained as remarkable as ever. His spirits were raised by his election to membership of the London Mathematical Society in December 1917, followed by fellowships of the Royal Society in May 1918 and Trinity College in October 1918. The war had enforced a prolonged stay in England, but by November 1918, his health had improved sufficiently for Hardy to write about a return to his homeland [5, p. 200]:

> He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is. His natural simplicity and modesty has never been affected in the least by success – indeed all that is wanted is to get him to realise that he really is a success.

On 27 February 1919, he embarked for India, arriving in Kumbakonam two weeks later, but his health deteriorated again despite medical treatment. He died on 26 April 1920 at the age of 32.

Ramanujan was described as being enthusiastic and eager, with a good-natured personality, although somewhat shy and quiet in official settings. Not particularly introspective, he was never able to give a completely coherent account of how he came up with his ideas – indeed, although his religious beliefs were later downplayed by Hardy (an atheist), there is evidence that Ramanujan believed that some form of divine inspiration was involved. In any case, he seems to have been quite modest about his own abilities and scrupulously keen to acknowledge help from any other sources. Littlewood famously remarked [2, p. xxxv] that 'every positive integer was one of his personal friends', and one of the best known stories told by Hardy appears to corroborate that opinion [2, p. xxxv]:

> I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number $(7 \cdot 13 \cdot 19)$ seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.'

With regard to his mathematics, its prime characteristic is its overwhelming wealth of algebraic formulae and vast computational complexity. Ramanujan was gifted with a power of calculation and symbolic dexterity unavailable to most mathematicians prior to the computer age. He also had an uncanny ability to spot patterns that nobody knew existed. For example, from the list of partition numbers from 1 to 200, he deduced a number of attractive, but hitherto unknown, congruences, including

$$p(5m+4) \equiv 0 \mod 5$$

and

$$p(25m+24) \equiv 0 \mod 25.$$

Most mathematicians would be satisfied with the mere discovery of relationships such as these, but in order to prove them Ramanujan was led to an even more stunning result

$$p(4) + p(9)x + p(14)x^2 + \cdots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6},$$

which is one of the most beautiful formulae he ever produced.

His ability to conjure up a myriad of bizarre yet almost supernaturally accurate approximations was another overwhelming feature of his mathematics. From his work on elliptic and modular functions came irrational expressions surprisingly close to integer values, such as

$$\mathrm{e}^{\pi\sqrt{58}} = 24\,591\,257\,751.999\,999\,822\ldots$$

It also yielded a series of tremendously accurate approximations to π, for example,

$$\frac{1}{2\pi\sqrt{2}} \approx \frac{1103}{99^2},$$

which is correct to 8 decimal places. But like so much of Ramanujan's mathematics, there is far more profound detail lying beneath the surface, since this approximation is merely the first term of an exact identity

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26\,390k)}{(k!)^4 396^{4k}},$$

which is itself intimately related to his earlier value of $\mathrm{e}^{\pi\sqrt{58}}$, since 26 390 is a multiple of 58, while $396^{4k} = (4^2 \times 99^2)^{2k}$, and

$$\mathrm{e}^{\pi\sqrt{58}} = 396^4 - 104.000\,000\,177\ldots$$

In the 100 years since his death, Ramanujan's mathematics has provided a constant source of inspiration and wonder for mathematicians. His work with Hardy on the partition function introduced a powerful new technique, subsequently developed by Hardy and Littlewood, now known as the 'circle method', which remains a valuable tool in additive number theory. The Hardy–Ramanujan partition formula itself was refined and improved by Hans Rademacher in the 1930s and, as well as its utility in mathematics, now serves as a useful function in superstring theory in physics and the study of phase transitions in chemistry. Ramanujan's manuscript notebooks – containing well over 3000 formulae and theorems – have been analysed intensively, with their contents being proved over the decades by subsequent generations of mathematicians. The discovery by George Andrews of a further 'lost notebook' in 1976 revealed a further 600 results. The mathematics contained within these notebooks, together with Ramanujan's published papers and innovations such as the Ramanujan theta function, Ramanujan primes and mock theta functions, have opened up new areas of mathematics and will no doubt continue to inspire and stimulate new mathematical ideas for many years to come.

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CLASSICS

Professor K G Ramanathan was one of India's leading expert on the great Indian mathematician Srinivsa Ramanujan and his work. work. He was responsible for recovering Ramanujan's notebooks from the archives of the University of Madras where they were collecting dust and decaying . He got them published by the TIFR with help from the Tata Trust. Later on an American mathematician George Andrews also uncovered Ramanujan's other notebooks from the library at Cambridge , England and got them published. Below is the story of the unpublished manuscripts of Srinivasa Ramanujan in KGR's own words.

K B Athreya

The Unpublished Manuscripts of Srinivasa Ramanujan

K G Ramanathan

Tata Institute of Fundamental Research, Bombay 400 005, India

Srinivasa Ramanujan, the greatest Indian mathematician of modern times, was born on December 22, 1887 at Erode in Tamil Nadu. After several unsuccessful attempts to pass the Intermediate examination of the Madras University, he joined, in February 1912, the Madras Port Trust as a grade 4 clerk on a monthly salary of rupees thirty. Even from the age of 18 or 19, Ramanujan began keeping his, now famous, Notebooks in which he recorded his discoveries. These Notebooks after several years of hibernation in the archives of the University of Madras were published and brought to the notice of the scientific public in 1957 by the Tata Institute of Fundamental Research, Bombay, with financial assistance from the Sir Dorabjee Tata Trust. Ramanujan's correspondence with G. H. Hardy of Cambridge, England, who was boundless in his appreciation of Ramanujan's mathematical work, resulted in his going to England in 1914 and staying there for five years during which he did work of "profound and invincible originality". In 1917 he fell seriously ill, partly because of the damp English climate and partly because of his reluctance – almost amounting to stubbornness – to take even vegetarian food which was not highly spiced. He was confined, for the greater part of his stay in England, to Nursing Homes and Sanatoria and returned to India early 1919, emaciated and in poor health. He passed away in Madras on April 26, 1920 hardly 33 years old.

Reproduced from *Current Science*, Vol.50, No.5, March 5, 1981, .
Thanks are due to the Master and Fellows of the Trinity College, Cambridge, England, for according us permission to print the three pages from Ramanujan's manuscript.

During his stay at the Sanatoria in England where he was most of the time confined to bed and in comparative solitude, he had been working on many problems related to the partition function and the 'tau' function and on general aspects of, what is now known as, 'Hecke Theory', in which he was indeed a pioneer. He seems to have begun there a long memoir (A) entitled "Congruence properties of $p(n)$ and $\tau\ (n)$ defined by the equations

$$\sum_{0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}$$

$$\sum_{1}^{\infty} \tau(n)x^n = x\left\{(1-x)\left(1-x^2\right)\times(1-x^3)\ldots\right\}^{24}."$$

He had also been working on various aspects of the continued fraction

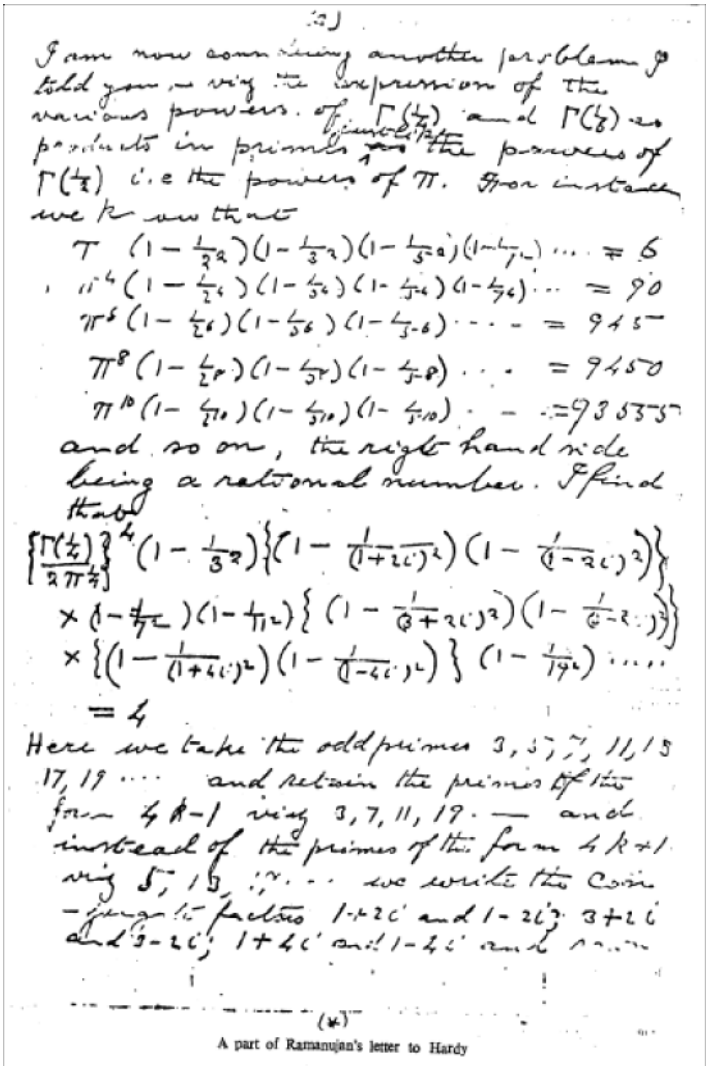
$$1+\frac{x}{1}+\frac{x^2}{1}+\frac{x^3}{1}+\ldots\ ,\ \ |x|<1$$

which was discovered independently by Rogers and Ramanujan. Its evaluation, however, for special values of x, is Ramanujan's own and has intimate connection with *complex multiplication*.

He had, in addition, been writing letters to Hardy fairly frequently. His letters were invariably full of mathematics and are very interesting. They deserve to be made public.

After his return to India in precarious health, he had been staying in Nursing Homes, especially in Madras. He seems to have done an enormous amount of work on q-series, continued fractions, etc. In fact he wrote a long letter to Hardy containing his results on mock-theta functions. He recorded his discoveries during this period in loose sheets of paper where, as was his wont, statement followed statement. A long manuscript, which is called by G. E. Andrews, the "Lost" notebook of Ramanujan has a strange history and seems to have been written during the last year of his life.

Shortly after Ramanujan's death in Madras on April 26, 1920, all his manuscripts were acquired by the University of Madras on payment of rupees



A part of Ramanujan's letter to Hardy

twenty per month for life to his widow. (It appears that this amount had been enhanced in the subsequent decades.) These manuscripts were later transmitted to G. H. Hardy. Since G. N. Watson of the University of Birmingham and B. M. Wilson intended to edit the Notebooks of Ramanujan, the manuscripts were entrusted with G. N. Watson. Hardy intended to edit the manuscript (A) mentioned above. He, however only published a short

note on some results of Ramanujan's in the manuscript (A). A few other results from (A) were supplied with proofs by J. M. Rushforth in his thesis in 1951 written with Watson. All the manuscripts remained with Watson and were, perhaps, forgotten by him. On Watson's death in 1965, these manuscripts were deposited by his widow, at the instance of R.A. Rankin of Glasgow University, with the Trinity College, Cambridge, of which Ramanujan was a Fellow. They were strangely found by G. E. Andrews[1] whose preprint made me aware of their existence at Trinity. The Tata Institute of Fundamental Research acquired photo copies of all these manuscripts in 1978.

Watson had also copied some manuscripts of Ramanujan (whose originals seem to be not available now) and these papers are now in the Oxford mathematical library. They were brought to light, a few years ago, by B. J. Birch[2].

To our knowledge, thus, there are the following Ramanujan manuscripts which are to be found in the libraries of the Trinity College, Cambridge and the Oxford mathematical library in England. Whether there are any other unpublished manuscripts of his, available elsewhere, we do not know.

MS (A) is the long memoir of 43 pages of foolscap-size on the congruence properties of $p(n)$ and $\tau(n)$. A detailed discussion of this was already given by us in ref. 6.

MS (B) are papers copied from loose sheets by Watson. They are 32 pages in length and are mentioned by Birch[2].

MS (C) Ramanujan's letters to Hardy written at various times while in Sanatoria and later from India.

MS (D) Miscellaneous notes available at Trinity College together with the edited, but unpublished, versions of some chapters of Ramanujan's Notebooks by B. M. Wilson.

MS (E) The so-called "Lost" Notebook of Ramanujan[1].

It would be difficult to discuss, in detail, the contents of these manuscripts. We shall, however, mention some of the interesting results in them which

might give a flavour of their contents. As stated above, we shall not discuss the manuscript (A).

MS (B) was discussed by Birch. He had published the results in (B) relating to Hecke theory and the identities connected with the functions $G(x)$ and $H(x)$ given by

$$G(x) = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \dots$$

$$H(x) = 1 + \frac{x^{1.2}}{1-x} + \frac{x^{2.3}}{(1-x)(1-x^2)} + \frac{x^{3.4}}{(1-x)(1-x^2)(1-x^3)} + \dots$$

In his note in the *Proceedings of the London Mathematical Society,* Records for March 1919 Ramanujan has given two identities between $H(x)$ and $G(x)$ and says "Each of these formulae is the simplest of a large class" Birch in his note[2] has published 40 such formulae from (B). It would be very interesting to give a systematic, and perhaps a unified, but not *ad hoc* proof of all these identities. Watson gave proofs of 9 of them[11].

The importance of manuscript (B) is however due to the large number of results on Dirichlet series with Euler product associated with cusp forms belonging to the modular group and its subgroups. One could justifiably say, that on the formal side, this is one of Ramanujan's most beautiful creations. In his only published paper[8], which, as Birch says, is one of the most beautiful published by the Cambridge Philosophical Society, Ramanujan gives Dirichlet series with Euler product. The interesting fact is that the p-factor of this Euler product is a quadratic irreducible polynomial. In (B) Ramanujan writes down many more examples of Euler products of this type related to subgroups of the modular group. There are many such examples in (D) and (E). It is really remarkable to see that Ramanujan with uncanny insight, chooses just those cusp forms whose associated Dirichlet series have Euler products. How he arrived at these cusp forms is a mystery. For instance, in (A) on page 28, he defines

$$\sum_1^\infty \Omega_2(n)x^n = x^2\{(1-x)(1-x^2)\dots\}^{48} \times \left(1 + 240\sum_1^\infty \frac{n^3x^n}{1-x^n}\right)$$

A page from the 'Lost' Notebook of Ramanujan

and

$$\sum_{1}^{\infty}\Omega_3(n)x^n = x^2\{(1-x)(1-x^2)\dots\}^{48} \times \left(1 - 504\sum_{1}^{\infty}\frac{n^5x^n}{1-x^n}\right)$$

and remarks: "The functions

$$\sum \frac{\Omega_2(n)}{n^3}, \sum \frac{\Omega_3(n)}{n^5}$$

are obviously not capable of a single product ... ; but they are, as a matter of fact, the differences of two such products." It is indeed a great tragedy that the war intervened during Ramanujan's stay in England and prevented him from meeting or even corresponding with E. Hecke in Germany who, as is well known, independently investigated 20 years later in 1937, the relationship between modular forms and Dirichlet series with Euler products[4]. He had many things in common with Hecke, at least on the formal side, which might have got exploited, had the two mathematicians (who were both born in 1887) met or corresponded.

Dr. S. S. Rangachari[10] has recently been able to elucidate the statements made in (B), (D) and (E) and relate them to the work of Hecke and others. As Rangachari shows, Ramanujan had been able to write down, just how one is unable to say, a basis of eigenfunctions (of Hecke operators) which alone possess Dirichlet series with Euler products. It is an astonishing piece of work by Ramanujan.

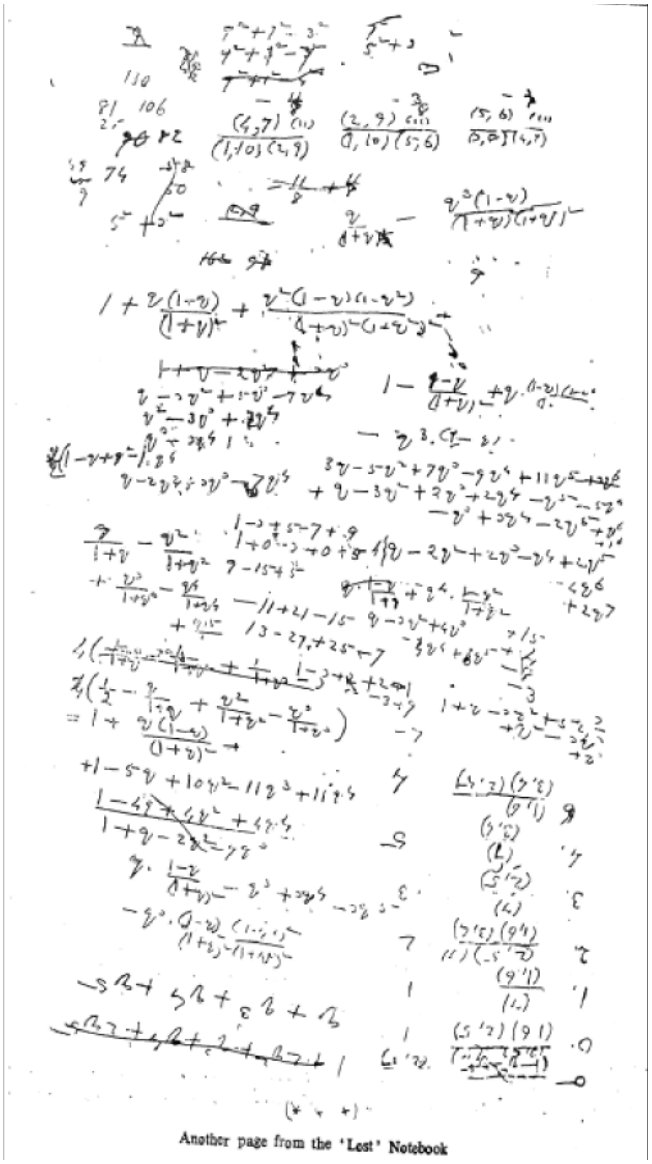
MS (C) contains a large number of letters related to asymptotic formulae for fourier coefficients of modular forms of positive dimension (negative weight). Ramanujan's letters deserves to be scrutinised further. There is one particular letter (as usual with no date) written in 1918 which is of interest. In it, Ramanujan evaluates series of the type

$$\begin{aligned} Z(s) = &\left(1-\tfrac{1}{3^s}\right)\left(1-\tfrac{1}{(1+2i)^s}\right) \\ &\times\left(1-\tfrac{1}{(1-2i)^s}\right)\left(1-\tfrac{1}{7^s}\right) \\ &\times\left(1-\tfrac{1}{11^s}\right)\left(1-\tfrac{1}{(3+2i)^s}\right) \\ &\times\left(1-\tfrac{1}{(3-2i)^s}\right)\ldots \end{aligned}$$

for s =2, 4, 6, 8, ... The numbers 3, $1+2i$, $1-2i$,7, 11, $3+2i$, $3-2i$, ... are all the 'odd' primes in the ring of Gaussian integers. Ramanujan says that

$$Z(s).\left(\frac{\Gamma\left(\frac{1}{4}\right)}{2\pi^{1/4}}\right)^{2s} = \lambda_s$$

is a rational number for s =2, 4, 6,... In fact $\lambda_2 = 4$, $\lambda_4 = 12$, ... This theme also occurs in (E). He evaluates λ_s for $s = 2, 4, \ldots 60$ and observes



Another page from the 'Lost' Notebook

that for $s = 4k$, they are related to Eienstein series

$$\sum_{a,b}{}'(a+ib)^{-4k}, \quad i = \sqrt{-1},$$

a and b running through all integers not simultaneously zero. Ramanujan also knows that this is related to the lemniscate integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

(see[9] for $k = 2$). In this case they are related to what are called Hurwitz numbers[5]. But Ramanujan's viewpoint is independent and goes much further. The series

$$(Z(s))^{-1} = \prod_p \left(1 - \frac{\chi(p)}{(Np)^s}\right)^{-1}$$

Where $\chi(p) = \bar{p}^s$, the bar denoting complex conjugation and $Np = p.\bar{p} > 0$ and p runs through the primes mentioned earlier, is a Hecke L-series with 'Grössencharaktere' related to the imaginary quadratic field $Q(\sqrt{-1})$. Curiously enough, Hecke was investigating such series precisely in 1918 in his researches on algebraic number theory[3,4]. Ramanujan's is a remarkable achievement and he writes to a letter Hardy, a part of which is reproduced on page 189.

A detailed discussion of these results is to be found in ref. 7.

MS (D) is a collection of unpublished parts of several papers which are published in the British periodicals during Ramanujan's stay in England. These papers had not been published in full presumably because of their length. For example, as Hardy mentions[8] (p.339), part of the long paper 'Highly composite Numbers' which was suppressed is to be found in (D) though some sections ((62)–(68)) are missing. It might be interesting to study these anew in the light of later work of P. Erdös and others.

A most interesting part of MS (D) is the section due to B.M. Wilson who had 'edited' twelve chapters from Ramanujan's Notebooks.

The most important manuscript, without doubt, is MS (E). It is a mine of formulae written in the style of the Notebooks containing a very large number of results on mock-theta functions, continued fractions, q-series, complex multiplication, etc. G. E. Andrews wrote to us last year that he

had proved a large number of formulae in this manuscript. We just quote one formula related to continued fractions; the formula is indicative of the fertility of his ideas and the beauty of his results:

$$\frac{2\sqrt{(x)}}{1} - \frac{1}{e^x + e^{-x}} - \frac{1}{e^{2x} + e^{-2x}} - \ldots = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} e^{G(x)}$$

where $G(x) = a_2 x^2 + a_4 x^4 + \ldots$ and

$$a_n = \frac{4\Gamma(n)}{\pi^{2n+1}} \left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots\right) \times \left(\frac{1}{1^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{+1n}} + \ldots\right)$$

$$a_2 = \frac{1}{48}, a_4 = \frac{1}{1152}, \ldots$$

It is to be noticed that Ramanujan knew not only the values of the zeta function at even positive integers but also of the L-series with real characters at odd positive integers. There is another example like the above in (E).

There are any number of variations related to Ramanujan's continued fraction mentioned earlier.

However, one of the most interesting features of the manuscript (E) is the insight, albeit slight, that it affords us into the way Ramanujan must have arrived at the beautiful results in (E). While Ramanujan had tremendous intuition about and insight into formulae, one does not *even now* know of the tremendous amount of computations and work that must have preceded before the many beautiful formulae crystallized. We reproduce (see pages 206 and 208) two facsimiles of "working". It is quite possible that the many papers in which he had done the "working" have either perished or been thrown away.

In conclusion, the fascinating life of Ramanujan and the beauty of the many results obtained by him in his several unpublished manuscripts encourages us to hope that many young and able mathematicians in India will study his work more carefully and relate Ramanujan's work to contemporary work in modular forms and arithmetic. Such work besides being greatly rewarding will be a signal service to the mathematical community.

CLASSICS

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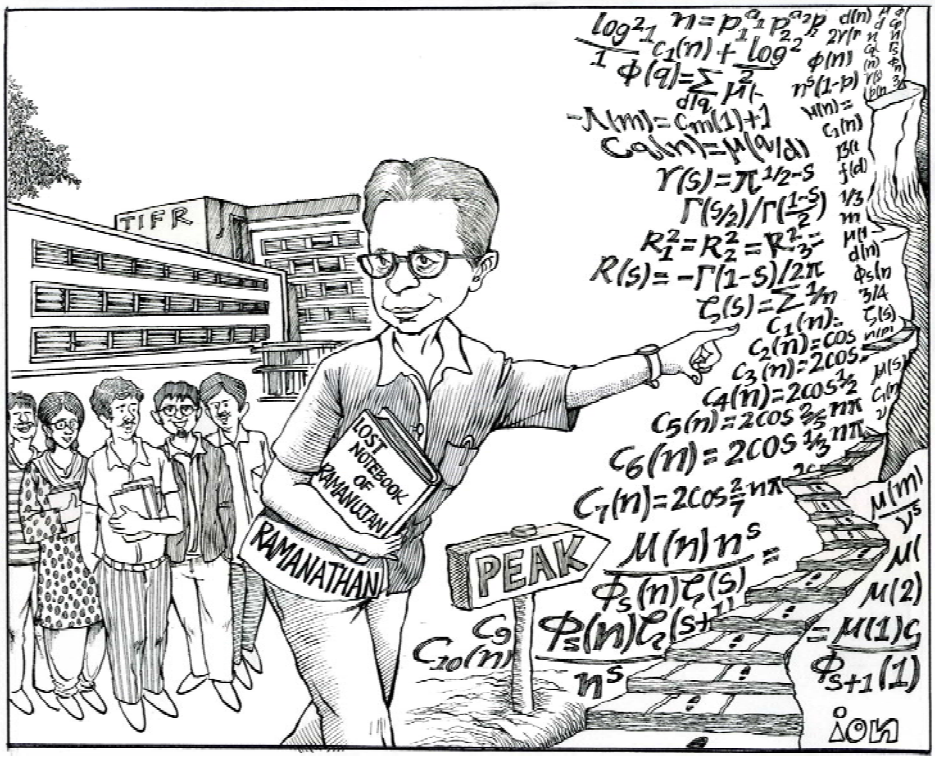
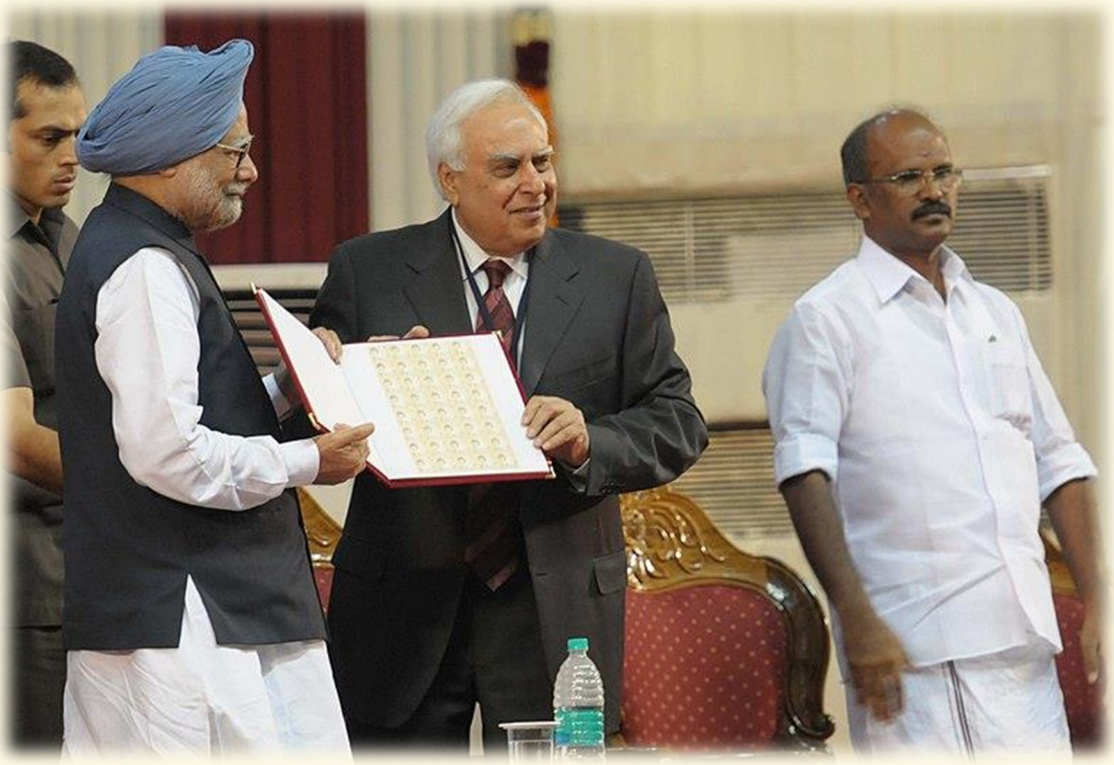


I shall be your guide to scale the peak of mathematical excellence.

Ramanujan with his Parents!



Manmohan Singh releasing the postal stamp on noted mathematician Shri Srinivas Ramanujan at his 125th Birth Anniversary celebrations.

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